YOKOHAMA MATHEMATICAL
JOURNAL VOL. 36, 1988

LOCAL SOLVABILITY OF FUCHSIAN ELLIPTIC EQUATIONS FOR HYPERFUNCTIONS

By

TOSHINORI ÔAKU

(Received August 15, 1988; Revised October 13, 1988)

Summary The local existence of a hyperfunction solution u to a linear partial differential equation $Pu=f$ is proved for any hyperfunction f when P is a degenerate elliptic operator of the Fuchs type.

1. Introduction

This paper is concerned with the local solvability of some degenerate linear elliptic partial differential equations for hyperfunctions. The local solvability at a point x^0 of $\boldsymbol{R}^n$ of the linear partial differential equation $Pu=f$ for hyperfunctions means that, for any hyperfunction f defined on a neighborhood of x^0, there exists a hyperfunction u on a (possibly smaller) neighborhood of x^0 satisfying $Pu=f$.

Local solvability for hyperfunctions is known in various cases: As the simplest case, the local solvability holds if P is elliptic. In fact, the fundamental theorem of Sato assures that the sheaf homomorphism

$$P: \mathcal{B}/\mathcal{A} \longrightarrow \mathcal{B}/\mathcal{A}$$

is an isomorphism, where $\mathcal{A}$ and $\mathcal{B}$ denote the sheaf on $\boldsymbol{R}^n$ of real analytic functions and that of hyperfunctions respectively. On the other hand, the homomorphism

$$P: \mathcal{A}_{x^0} \longrightarrow \mathcal{A}_{x^0}$$

is surjective by virtue of the Cauchy-Kowalewsky theorem, where $\mathcal{A}_{x^0}$ denotes the stalk at x^0 of the sheaf $\mathcal{A}$. Hence we get the surjectivity of the homomorphism

$$P: \mathcal{B}_{x^0} \longrightarrow \mathcal{B}_{x^0},$$

which means the local solvability of $Pu=f$ for hyperfunctions.

A more general result has been obtained by Kashiwara-Kawai (Theorem 6.5 of [5]): Assume that, for any point $x^*=(x^0, \sqrt{-1}\xi)$ of the purely imaginary cosphere bundle $\sqrt{-1}S^*\boldsymbol{R}^n$, there exists $\eta\in\boldsymbol{R}^n$ such that P is partially micro-hyperbolic in the direction $\langle \eta, dx\rangle$ (in the sense of [5]). Then the equation $Pu=f$ for hyperfunctions is locally solvable at x^0.

This condition of Kashiwara-Kawai does not hold for degenerate equations such as Fuchsian equations in the sense of Baouendi-Goulaouic [1] (we shall recall the definition of Fuchsian equations in Section 2). The local solvability of Fuchsian hyperbolic equations for hyperfunctions was obtained by Tahara as a consequence of his extensive and elaborate work on Fuchsian equations (Theorem 2.3.6 of [7]). It seems, however, that the local solvability of Fuchsian elliptic equations has not been proved yet, which is our main aim.

The motivation of this paper comes from the stimulus discussion with Prof. C. Parenti and Prof. H. Tahara.

2. Main results.

Let P be a linear partial differential operator of order m with real analytic coefficients defined on an open neighborhood U of a point x^0 of $\boldsymbol{R}^n$. We assume that there exists a real-valued real analytic function φ on U with $\varphi(x^0)=0$ and $d\varphi\neq 0$ such that the principal symbol $\sigma_m(P)$ of P is written in the form

$$\sigma_m(P)(x, \xi)=\varphi(x)^k p_m(x, \xi)$$

with an integer $k\geqq 1$, where $p_m(x, \xi)$ is a real analytic function on $U\times\boldsymbol{R}^n$ such that $p_m(x, \xi)\neq 0$ for any $x\in U$ and $\xi\in\boldsymbol{R}^n-\{0\}$.

Theorem 1. *Under the above assumptions, the homomorphism*

$$P: (\mathcal{B}/\mathcal{A})_{x^0} \longrightarrow (\mathcal{B}/\mathcal{A})_{x^0}$$

is surjective; i.e., $Pu=f$ is locally solvable for the sheaf $\mathcal{B}/\mathcal{A}$.

We shall prove this theorem in Section 3. As an immediate consequence of this theorem and the exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{A} \longrightarrow 0,$$

we get the following:

Corollary. *Under the same assumptions as Theorem* 1, *the homomorphism*

$$P: \mathcal{B}_{x^0} \longrightarrow \mathcal{B}_{x^0}$$

is surjective if so is the homomorphism

$$P: \mathcal{A}_{x^0} \longrightarrow \mathcal{A}_{x^0}.$$

Now we recall the definition of Fuchsian partial differential operator after

Baouendi-Goulaouic [1]. Let S be the hypersurface in U defined by $\varphi=0$. Let $x=(x_1, \cdots, x_n)$ be a real analytic local coordinate system around x^0 such that $x_1=\varphi$ and $x=0$ at x^0. We use the notation $x'=(x_2, \cdots, x_n)$, $D_j=\partial/\partial x_j$ $(j=1, \cdots, n)$, $D'=(D_2, \cdots, D_n)$. Then P is said to be a Fuchsian operator of weight $m-k$ with respect to S (around x^0) if $k\leqq m$ and if, on a neighborhood of x^0, P is written in the form

$$P=a(x)(x_1^k D_1^m+A_1(x, D')x_1^{k-1}D_1^{m-1}+\cdots$$
$$+A_k(x, D')D_1^{m-k}+\cdots+A_m(x, D'))\,,$$

where $A_j(x, D')$ is a linear partial differential operator of order at most j for $j=1, \cdots, m$; $A_j(0, x', D')$ is of order 0, i.e. equals a function $a_j(x')$ for $j=1, \cdots, k$; $a(x')$ is a real analytic function with $a(x^0)\neq 0$. Then the non-trivial characteristic exponents $\lambda_j(x^0)$ $(j=1, \cdots, k)$ of P at x^0 are defined as the roots of the equation in λ:

$$(\lambda-m+k)(\lambda-m+k-1)\cdots(\lambda-m+1)+a_1(0)(\lambda-m+k)(\lambda-m+k-1)$$
$$\cdots(\lambda-m+2)+\cdots+a_{k-1}(0)(\lambda-m+k)+a_k(0)=0\,.$$

Note that the condition of Fuchsian operator and the non-trivial characteristic exponents are independent of the choice of the defining function φ of S.

Theorem 2. *Assume P satisfies the same assumptions as Theorem 1. Assume moreover that P is a Fuchsian operator of weight $m-k$ with respect to S, and that $\lambda_j(x^0)$ is not equal to an integer $\geqq m-k$ for $j=1, \cdots, k$. Then the homomorphism*

$$P:\ \mathcal{B}_{x^0} \longrightarrow \mathcal{B}_{x^0}$$

is surjective.

Proof. Under these conditions, by Theorem 1 of [1], $Pu=f$ is locally solvable at x^0 for real analytic functions. Hence we get the local solvability for hyperfunctions from Corollary.

3. Proof of Theorem 1

In this section we always assume the assumptions of Theorem 1. We may also assume $\varphi=x_1$ and $x^0=0$ since we are concerned with local properties. We investigate the equation $Pu=f$ for microfunctions. For this purpose we use the language of the sheaf cohomology theory systematically (see, e.g., Bredon [3]).

We denote by $\mathcal{C}$ the sheaf of microfunctions on the purely imaginary cosphere bundle $\sqrt{-1}S^*\boldsymbol{R}^n=\boldsymbol{R}^n\times\sqrt{-1}S^{n-1}$ (S^{n-1} denotes the $(n-1)$-dimensional unit sphere), and by π the natural projection of $\sqrt{-1}S^*\boldsymbol{R}^n$ to $\boldsymbol{R}^n$. We use the notation $\xi'=(\xi_2, \cdots, \xi_n)$ for $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ and put

$$\Xi=\{(x, \sqrt{-1}\xi)\in\sqrt{-1}S^*U\,;\,\xi'\neq 0\},$$

$$\Xi_0=\{(x, \sqrt{-1}\xi)\in\Xi\,;\,x_1=0\}.$$

Then P is an operator of constant multiplicity and its bicharacteristics are the fibers of the map

$$\rho:\Xi_0\longrightarrow\sqrt{-1}S^*U_0=U_0\times\sqrt{-1}S^{n-2}$$

defined by $\rho(0, x', \sqrt{-1}\xi)=(x', \sqrt{-1}\xi'/|\xi'|)$, where $U_0=\{x\in U\,;\,x_1=0\}$. Hence $Pu=f$ is equivalent to a partial de Rham system (see Sato-Kawai-Kashiwara [6]). Let us denote by $\mathcal{S}$ the sheaf of the microfunction solutions of the equation $Pu=0$. Then in view of [6, Chapater III] we have an exact sequence of sheaves

$$0\longrightarrow\mathcal{S}\longrightarrow\mathcal{C}\xrightarrow{P}\mathcal{C}\longrightarrow 0 \tag{1}$$

on Ξ, which means that the equation $Pu=f$ is (micro-)locally solvable for microfunctions. Moreover, $\mathcal{S}$ is supported by Ξ_0 and is isomorphic to the inverse image $\rho^{-1}\mathcal{C}'$ of the sheaf $\mathcal{C}'$ of microfunctions on $\sqrt{-1}S^*U_0$.

Lemma. *Let V be an open subset of Ξ. Then the homomorphism*

$$P:\mathcal{C}(V)\longrightarrow\mathcal{C}(V)$$

is surjective if and only if the 1-*st cohomology group $H^1(V, \mathcal{S})$ vanishies.*

Proof. From the short exact sequence (1), we get the long exact sequence

$$0\longrightarrow\mathcal{S}(V)\longrightarrow\mathcal{C}(V)\xrightarrow{P}\mathcal{C}(V)\longrightarrow H^1(V, \mathcal{S})\longrightarrow H^1(V, \mathcal{C})=0$$

since $\mathcal{C}$ is a flabby sheaf. This completes the proof.

For $r>0$, put $K_r=\{(x, \sqrt{-1}\xi)\in\Xi\,;\,|\xi_1|\leqq r|\xi'|\}$. Then we first show the following

Proposition. *For any $r>0$ and for any open subset U' of U, the homomorphism*

$$P:\mathcal{C}(K_r\cap\pi^{-1}(U'))\longrightarrow\mathcal{C}(K_r\cap\pi^{-1}(U'))$$

is surjective.

Proof. First note that we may assume $U'=U$. We use the notion and the notation of derived categories (cf. Hartshorne [4]). We denote by $\Gamma$the functor taking the global sections of sheaves, and by $\boldsymbol{R}\Gamma$ its derived functor. Then we get quasi-isomorphisms

$$(2) \qquad \boldsymbol{R}\Gamma(K_r, \mathcal{S})=\boldsymbol{R}\Gamma(K_r, \rho^{-1}\mathcal{C}')=\boldsymbol{R}\Gamma(\rho(K_r\cap\Xi_0), \boldsymbol{R}\rho_*\rho^{-1}\mathcal{C}'),$$

where we denote the restriction of ρ to $K_r\cap\Xi_0$ by the same letter ρ, denote by ρ_* the functor of direct image, and by $\boldsymbol{R}\rho_*$ its derived functor. Since the map ρ restricted to $K_r\cap\Xi_0$ is a proper map with contractible fibers, we have

$$(R^j\rho_*\rho^{-1}\mathcal{C}')_{y^*}=H^j(\rho^{-1}(y^*), \rho^{-1}\mathcal{C}')=0$$

for any $j\geqq 1$ and $y^*\in\sqrt{-1}S^*U_0$ since, restricted to $\rho^{-1}(y^*)$, $\rho^{-1}\mathcal{C}'$ is a constant sheaf (cf. [3]). Hence we get

$$(3) \qquad \boldsymbol{R}\rho_*\rho^{-1}\mathcal{C}'=\rho_*\rho^{-1}\mathcal{C}'=\mathcal{C}'.$$

Combining (2) and (3) we have

$$H^1(K_r, \mathcal{S})=H^1(\rho(K_r\cap\Xi_0), \mathcal{C}')=0$$

since $\mathcal{C}'$ is flabby. In view of Lemma, this completes the proof of Proposition.

Now let us prove Theorem 1. Let f be a section of $\mathcal{B}/\mathcal{A}=\pi_*\mathcal{C}$ on a neighborhood of x^0. We may assume that f is defined on U. Hence f can be regarded as a section of $\mathcal{C}$ on $\sqrt{-S}^*U$. In view of Proposition, there exists a section u of $\mathcal{C}$ on a neighborhood of $K=K_1$ satisfying $Pu=f$ there.

Since $\mathcal{C}$ is flabby we can find a section $\tilde{u}$ of $\mathcal{C}$ on $\sqrt{-1}S^*U$ which coincides with u on a neighborhood of K. Then $g=P\tilde{u}-f$ is a section of $\mathcal{C}$ on $\sqrt{-1}S^*U$ whose support does not meet K.

There exist a complex neighborhood $\tilde{U}$ of U in $\boldsymbol{C}^n$ and holomorphic functions G_+ and G_- defined on $\Gamma_+\cap\tilde{U}$ and on $\Gamma_-\cap\tilde{U}$ respectively such that

$$g(x)=\mathrm{sp}(G_+(x+\sqrt{-1}\Gamma_+0))+\mathrm{sp}(G_-(x+\sqrt{-1}\Gamma_-0)).$$

Here $\Gamma_\pm=\{y=(y_1, \cdots, y_n)\in\boldsymbol{R}^n; \pm y_1>|y'|\}$; $G_+(x+\sqrt{-1}\Gamma_+0)$ means the hyperfunction defined as the boundary value of G_+ as $y=\mathrm{Im}\, z$ tends to 0 with $y\in\Gamma_+$; sp denotes the spectral map of $\mathcal{B}$ to $\pi_*\mathcal{C}$.

Since $p_m(x^0, \xi)\neq 0$ for any $\xi\in\boldsymbol{R}^n-\{0\}$, we can take a sufficiently small $\varepsilon>0$ so that $p_m(z, \xi)\neq 0$ if $z\in\boldsymbol{C}^n$, $|z|<\varepsilon$ and $\xi\in\boldsymbol{R}^n-\{0\}$. Note that the polar of the cone $\boldsymbol{R}^n+\sqrt{-1}\Gamma_+$ satisfies

$$\begin{aligned}(\boldsymbol{R}^n+\sqrt{-1}\Gamma_+)^\circ&=\{\zeta\in\boldsymbol{C}^n; \mathrm{Re}\langle z, \zeta\rangle\leqq 0 \text{ if } \mathrm{Im}\, z\in\Gamma_+\}\\ &\subset\{\sqrt{-1}\eta; \eta=(\eta_1, \eta')\in\boldsymbol{R}^n, \eta_1\geqq|\eta'|\}.\end{aligned}$$

Then by Théorème 2.1 and the proof of Lemma 3.2 of Bony-Schapira [2], we can find holomorphic functions $F_\pm$ on $\{z\in\boldsymbol{C}^n; |z|<\delta, \mathrm{Im}\, z\in\Gamma_\pm\}$ with some $\delta>0$ smaller than ε such that $PF_\pm=G_\pm$. (Although P itself does not satisfy the assumption of Lemme 3.2 of [2], it is easy to verify the existence of $F_\pm$ by the same argument using Théorème 2.1 of [2] noting that $z_1 \neq 0$ if $\operatorname{Im} z \in \Gamma_{\pm}$. This fact was pointed out by H. Tahara in his master's thesis, University of Tokyo, 1975.) Put

$$v = \mathrm{sp}(F_+(x+\sqrt{-1}\Gamma_+0)) + \mathrm{sp}(F_-(x+\sqrt{-1}\Gamma_-0)).$$

Then $u = \tilde{u} - v$ is a section of $\mathcal{C}$ on $\{(x, \sqrt{-1}\xi) \in S^*U\,;\, |x| < \delta\}$ and satisfies

$$Pu = P\tilde{u} - Pv = (f+g) - g = f.$$

This completes the proof of Theorem 1.

References

[1] Baouendi, M.S. and Goulaouic, C.: *Cauchy problems with characteristic initial hypersurface,* Comm. Pure Appl. Math. **26** (1973), 455–475.

[2] Bony, J.-M. and Schapira, P.: *Existence et prolongement des solutions holomorphes des équations aux dérivées partielles,* Invent. Math. **17** (1972), 95–105.

[3] Bredon, G.E.: *Sheaf Theory,* McGraw-Hill, New York, 1966.

[4] Hartshorne, R.: *Residues and Duality,* Lecture Notes in Mathematics No. **20**, Springer, Berlin, 1966.

[5] Kashiwara, M. and Kawai, T.: *Micro-hyperbolic pseudo-differential operators I,* J. Math. Soc. Japan **27** (1975), 359–404.

[6] Sato, M., Kawai, T. and Kashiwara, M.: *Microfunctions and pseudo-differential equations,* Lecture Notes in Mathematics No. **287**, pp. 265–529, Springer, Berlin, Heidelberg, New York, 1973.

[7] Tahara, H.: *Fuchsian type equations and Fuchsian hyperbolic equations,* Japan. J. Math. **5** (1979), 245–347.

Department of Mathematics
Yokohama City University
22-2 Seto, Kanazawa-ku
Yokohama, 236 Japan