Local Bernstein-Sato ideals: An algorithm and examples

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Abstract

Let $\mathbf{k}$ be a field of characteristic 0. For a polynomial mapping $f = (f_1, \dots, f_p)$ of $\mathbf{k}^n$ to $\mathbf{k}^p$, the local Bernstein-Sato ideal of f at a point $a \in \mathbf{k}^n$ is defined as an ideal of the ring of polynomials in $s = (s_1, \dots, s_p)$. We propose an algorithm for computing local Bernstein-Sato ideals combining Gröbner bases in rings of differential operators and primary decomposition in a polynomial ring. It also enables us to compute a constructible stratification of $\mathbf{k}^n$ such that the local Bernstein-Sato ideal is constant along each stratum. We also present examples, some of which have non-principal Bernstein-Sato ideals, computed with our algorithm by using a computer algebra system Risa/Asir.

Key words: Bernstein-Sato ideal, D-module, Gröbner base, primary decomposition

Introduction

Let n be a positive integer, $\mathbf{k}$ a field of characteristic zero, and $a = (a_1, \dots, a_n)$ a fixed point in $\mathbf{k}^n$. Let $x = (x_1, \dots, x_n)$ be a set of indeterminates. In this introduction, A shall be one of the following rings: the polynomial ring $\mathbf{k}[x]$; the localization $\mathbf{k}[x]_a$ of $\mathbf{k}[x]$ at a; the formal power series ring $\hat{\mathcal{O}}_{\mathbf{k}^n,a} = \mathbf{k}[[x - a]] = \mathbf{k}[[x_1 - a_1, \dots, x_n - a_n]]$; and when $\mathbf{k} = \mathbb{C}$, the ring $\mathcal{O}_{\mathbb{C}^n,a}$ of germs of complex analytic functions at a. Denote by ∂_{x_i}

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the partial differential operator $\frac{\partial}{\partial x_i}$ and by $D_A = A\langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle$ the ring of differential operators with coefficients in A.

Let $p \geq 1$ be an integer and let us consider $f = (f_1, \ldots, f_p) \in A^p$. Denote by F the product $f_1 \cdots f_p$ and let us introduce a set of indeterminates $s = (s_1, \ldots, s_p)$ and the $A[1/F, s]$-free module

$$\mathcal{L}_A = A[1/F, s] \cdot f^s$$

with $f^s = f_1^{s_1} \cdots f_p^{s_p}$. The set $\mathcal{L}_A$ is naturally endowed with a $D_A[s]$-module structure. Indeed, given $g \in A[1/F, s]$, we have

$$\partial_{x_i} \cdot g f^s = \Big(\frac{\partial g}{\partial x_i} + g \sum_{j=1}^{p} s_j \frac{\partial f_j}{\partial x_i} f_j^{-1} \Big) f^s.$$

The module $\mathcal{L}_A$ is an interesting and important object not only in D-module theory but also in, e.g., algebraic geometry, the theory of prehomogeneous vector spaces, and the theory of hypergeometric functions in several variables. For example, Oaku and Takayama (1999) proposed an algorithm for computing the twisted de Rham cohomology groups of the complement of the affine hypersurface $F = 0$ in $\mathbb{C}^n$ by using the D-module structure of $\mathcal{L}_A$.

The *Bernstein-Sato ideal* of f (with respect to A) is defined to be the ideal

$$\mathcal{B}_A(f) = \{ b(s) \in \mathbf{k}[s] \mid b(s) f^s \in D_A[s] \cdot F f^s \}$$

of $\mathbf{k}[s]$ and plays an essential role in studying the D-module structure of $\mathcal{L}_A$. If $p = 1$, the monic generator of $\mathcal{B}_A(f)$ is called the Bernstein-Sato polynomial of f (with respect to A) (see (Bernstein, 1972)). When $f \in \mathbf{k}[x]^p$, $\mathcal{B}_{\mathbf{k}[x]}(f)$ is called the *global Bernstein-Sato ideal* and $\mathcal{B}_{\mathbf{k}[x]_a}(f)$ is called the *local Bernstein-Sato ideal* at $a \in \mathbf{k}^n$. It is easy to see that $\mathcal{B}_{\mathbf{k}[x]_a}(f)$ is equal to $\mathcal{B}_{\hat{\mathcal{O}}_{\mathbf{k}^n,a}}(f)$ and that $\mathcal{B}_{\mathbb{C}[x]_a}(f)$ is equal to $\mathcal{B}_{\mathcal{O}_{\mathbb{C}^n,a}}(f)$ if $f \in \mathbb{C}[x]^p$. When $f \in \mathcal{O}_{\mathbb{C}^n,a}$, $\mathcal{B}_{\mathcal{O}_{\mathbb{C}^n,a}}(f)$ is called the analytic Bernstein-Sato ideal of f, and it is equal to $\mathcal{B}_{\hat{\mathcal{O}}_{\mathbb{C}^n,a}}(f)$. Finally, when $f \in \hat{\mathcal{O}}_{\mathbf{k}^n,a}$, we call $\mathcal{B}_{\hat{\mathcal{O}}_{\mathbf{k}^n,a}}(f)$ the formal Bernstein-Sato ideal of f (at a).

It was proved by Sabbah (1987) that analytic Bernstein-Sato ideals are not zero. See also (Bahloul, 2005a) for a constructive proof. Theoretical studies of Bernstein-Sato ideals can also be found in, e.g., (Maynadier, 1997), (Briançon and Maynadier, 1999), (Briançon and Maisonobe, 2002), (Bahloul, 2005b).

For a polynomial mapping, a general algorithm for the global Bernstein-Sato ideal was first proposed by Oaku and Takayama (1999); its modifications have been given by Bahloul (2001), Briançon and Maisonobe (2002), Levandovskyy and Morales (2008). On the other hand, for $p = 1$, Oaku (1997a) gave an algorithm for the local Bernstein-Sato polynomial at a given point (see also the recent work by Nakayama (2009)).

The first goal of the present paper is to present an algorithm for computing $\mathcal{B}_{\mathbf{k}[x]_a}(f)$ for a given $f \in \mathbf{k}[x]^p$ with $p \geq 1$ and $a \in \mathbf{k}^n$. For this purpose, we combine the algorithm of Oaku and Takayama (1999) for the global Bernstein-Sato ideal, which is based on Gröbner base computations in rings of differential operators, with primary decomposition in a polynomial ring, in the same way as was proposed by Oaku (1997b) in the case $p = 1$. This algorithm also provides us with a constructible stratification of $\mathbf{k}^n$ such that for a running over a given stratum the local Bernstein-Sato ideal at a is constant. The existence of such a stratification was proved theoretically by Briançon and Maisonobe (2002).

We have implemented our algorithm in a computer algebra system Risa/Asir (Noro el al.). Experimentation suggests that, at least in 'simple' cases, global and local Bernstein-Sato ideals are mostly principal, i.e., generated by a single element. In fact, Maynadier (1997) proved that the local Bernstein-Sato ideal is principal if $n = p = 2$ and $f = (f_1, f_2)$ defines a quasi-homogeneous complete intersection with an isolated singularity. On the other hand, Briançon and Maynadier (1999) showed that the local Bernstein-Sato ideal of $f = (z, x^4 + y^4 + 2zx^2y^2)$ in three variables (x, y, z) at the origin is not principal without giving its explicit generators. We can compute the two generators of it using our algorithm (see Example 3). Rather surprisingly, the global Bernstein-Sato ideal of the same f *is* principal. This exemplifies the importance of computing the *local* Bernstein-Sato ideal. We also present some variants of this example.

In Section 1, we describe fundamental properties and the algorithm. For the sake of clarity, all the proofs are postponed to Section 2. In Section 3, we give some examples computed with our algorithm (over the rationals) together with a validity proof of the results over the complex numbers. Finally in Section 4, we give some remarks on our implementation in Risa/Asir.

1. An algorithm for local Bernstein-Sato ideals

Let us fix a polynomial mapping $f = (f_1, \ldots, f_p) \in \mathbf{k}[x]^p$. We are interested in $\mathcal{B}_{\mathbf{k}[x]_a}(f)$. As we recalled, the formal, analytic (if $\mathbf{k} = \mathbb{C}$), and local Bernstein-Sato ideals of f at a are the same. So we shall use the notation $\mathcal{B}_{loc,a}(f) = \mathcal{B}_{\mathbf{k}[x]_a}(f)$, which shall be contrasted with the global Bernstein-Sato ideal $\mathcal{B}_{glob}(f) = \mathcal{B}_{\mathbf{k}[x]}(f)$.

Moreover, we shall use the notations $D = \mathbf{k}[x]\langle\partial_x\rangle$, $D_a = \mathbf{k}[x]_a\langle\partial_x\rangle$, $\hat{D}_a = \hat{\mathcal{O}}_{\mathbf{k}^n,a}\langle\partial_x\rangle$, and when $\mathbf{k} = \mathbb{C}$, $\mathcal{D}_a = \mathcal{O}_{\mathbb{C}^n,a}\langle\partial_x\rangle$.

Following Malgrange (1974), let us introduce new variables $t = (t_1, \ldots, t_p)$ together with the associated partial derivation operators $\partial_t = (\partial_{t_1}, \ldots, \partial_{t_p})$ and consider the ring $\hat{D}_a\langle t, \partial_t\rangle = \hat{D}_a \otimes_{\mathbf{k}} \mathbf{k}[t]\langle\partial_t\rangle$. We also consider subrings $D\langle t, \partial_t\rangle$, $D_a\langle t, \partial_t\rangle$ and (when $\mathbf{k} = \mathbb{C}$) $\mathcal{D}_a\langle t, \partial_t\rangle$.

The free module $\mathcal{L}_{\hat{\mathcal{O}}_{\mathbf{k}^n,a}} = \hat{\mathcal{O}}_{\mathbf{k}^n,a}[1/F, s]f^s$ has a $\hat{D}_a\langle t, \partial_t\rangle$-module structure defined by

$$\begin{aligned} t_j \cdot g(s)f^s &= g(s_1, \ldots, s_j + 1, \ldots, s_p)f_j f^s, \\ \partial_{t_j} \cdot g(s)f^s &= -s_j g(s_1, \ldots, s_j - 1, \ldots, s_p)f_j^{-1} f^s \end{aligned}$$

for $g(s) \in \hat{\mathcal{O}}_{\mathbf{k}^n,a}[1/F, s]$. It follows that $-\partial_{t_j} t_j$ acts on $\mathcal{L}_{\hat{\mathcal{O}}_{\mathbf{k}^n,a}}$ as s_j. Thus we shall identify s_j with $-\partial_{t_j} t_j$ and the rings $D[s]$, $D_a[s]$, $\mathcal{D}_a[s]$ and $\hat{D}_a[s]$ shall be regarded as subrings of $\hat{D}_a\langle t, \partial_t\rangle$.

Let us consider the following $p + n$ elements of $D\langle t, \partial_t\rangle$:

$$t_j - f_j \quad (j = 1, \ldots, p), \quad \partial_{x_i} + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial_{t_j} \quad (i = 1, \ldots, n). \tag{1}$$

One can easily check that these elements annihilate f^s. In fact we have

Lemma 1. *The annihilating ideals*[1] $\mathrm{ann}_{\hat{D}_a\langle t,\partial_t\rangle}(f^s) = \{P \in \hat{D}_a\langle t,\partial_t\rangle \mid Pf^s = 0\}$, $\mathrm{ann}_{\mathcal{D}_a\langle t,\partial_t\rangle}(f^s)$, $\mathrm{ann}_{D_a\langle t,\partial_t\rangle}(f^s)$, $\mathrm{ann}_{D\langle t,\partial_t\rangle}(f^s)$ *are all generated by the elements in (1).*

Now we introduce the following ideals:

- $I = \mathrm{ann}_{D\langle t,\partial_t\rangle} f^s, \quad J = \mathrm{ann}_{D_a\langle t,\partial_t\rangle} f^s$.
- $I_1 = \mathrm{ann}_{D[s]} f^s = I \cap D[s] \subset D[s], \quad J_1 = \mathrm{ann}_{D_a[s]} f^s = J \cap D_a[s] \subset D_a[s]$.
- $I_2 = (I_1 + D[s] \cdot F) \cap \mathbf{k}[x,s] \subset \mathbf{k}[x_1,\ldots,x_n,s_1,\ldots,s_p]$,
  $J_2 = (J_1 + D_a[s] \cdot F) \cap \mathbf{k}[x]_a[s] \subset \mathbf{k}[x_1,\ldots,x_n]_a[s_1,\ldots,s_p]$.
- $I_3 = I_2 \cap \mathbf{k}[s] \subset \mathbf{k}[s_1,\ldots,s_p], \quad J_3 = J_2 \cap \mathbf{k}[s] \subset \mathbf{k}[s_1,\ldots,s_p]$.

Proposition 2. $I_3 = \mathcal{B}_{glob}(f)$ *and* $J_3 = \mathcal{B}_{loc,a}(f)$.

In both global and local cases, we start with the 'same' ideals I and J in the sense that they admit a common set of generators. Then we construct in parallel the ideals I_k and J_k with $k = 1,2,3$ to get the global Bernstein-Sato ideal $I_3 = \mathcal{B}_{glob}(f)$ and the local Bernstein-Sato ideal $J_3 = \mathcal{B}_{loc,a}(f)$ respectively. It is natural to ask whether I_k and J_k are the same (in the above sense). Here is the beginning of the answer.

Proposition 3. $J_1 = D_a[s] \cdot I_1$ *and* $J_2 = \mathbf{k}[x]_a[s] \cdot I_2$.

This proposition implies that the global and the local constructions coincide up to I_2 and J_2. The passage from I_2 to I_3 consists in the usual elimination of x variables. However, the passage from J_2 to J_3 is different:

Proposition 4. *Let* Υ *be an ideal in* $\mathbf{k}[x,s]$ *and* a *be a point of* $\mathbf{k}^n$. *Let* $\Upsilon = \Upsilon_1 \cap \cdots \cap \Upsilon_r$ *be a primary decomposition of* Υ. *Set*

$$\sigma_a = \{i \in \{1,\ldots,r\} \mid a \in V(\Upsilon_i \cap \mathbf{k}[x])\},$$

where $V(\cdot)$ *stands for 'the zero set of'. Then we have*

$$(\mathbf{k}[x]_a[s] \cdot \Upsilon) \cap \mathbf{k}[s] = (\bigcap_{i\in\sigma_a} \Upsilon_i) \cap \mathbf{k}[s]$$

with the equality $\bigcap_{i\in\sigma_a} \Upsilon_i = \mathbf{k}[x,s]$ *if* $\sigma_a = \emptyset$.

So far, a was a fixed point in $\mathbf{k}^n$. Now we are concerned with the behaviour of $\mathcal{B}_{loc,a}(f)$ when a runs over $\mathbf{k}^n$. Let us apply Proposition 4 to a primary decomposition $\Upsilon_1 \cap \cdots \cap \Upsilon_r$ of I_2 to obtain the following:

Corollary 5. *For each subset* $\sigma \subset \{1,\ldots,r\}$, *set*

- $W_\sigma = \mathbf{k}^n \setminus (\bigcup_{i=1}^r V(\Upsilon_i \cap \mathbf{k}[x]))$ *if* $\sigma = \emptyset$,
- $W_\sigma = \bigcap_{i=1}^r V(\Upsilon_i \cap \mathbf{k}[x])$ *if* $\sigma = \{1,\ldots,r\}$,
- $W_\sigma = (\bigcap_{i\in\sigma} V(\Upsilon_i \cap \mathbf{k}[x])) \setminus (\bigcup_{i\notin\sigma} V(\Upsilon_i \cap \mathbf{k}[x]))$ *otherwise.*

Then $\bigcup_\sigma W_\sigma$ *is a constructible stratification of* $\mathbf{k}^n$ *such that the map* $\mathbf{k}^n \ni a \mapsto \mathcal{B}_{loc,a}(f)$ *is constant on each* W_σ.

Summed up, our algorithm is described as follows:

Algorithm 1.

[1] In a non commutative ring, ideal shall always mean left ideal.

Input $f = (f_1, \dots, f_p) \in \mathbf{k}[x]^p$ and $a \in \mathbf{k}^n$.

Step 1 Compute the annihilator $I_1 = \mathrm{ann}_{D[s]} f^s$ as follows:

(i) Introducing indeterminates $u = (u_1, \dots, u_p)$ and $v = (v_1, \dots, v_p)$, let $\tilde{I}$ be the left ideal of $D\langle t, \partial_t \rangle[u, v]$ generated by

$$t_j - u_j f_j,\ 1 - u_j v_j \quad (j = 1, \dots, p), \quad \partial_{x_i} + \sum_{j=1}^{p} u_j \frac{\partial f_j}{\partial x_i} \partial_{t_j} \quad (\ i = 1, \dots, n).$$

(ii) Let $\tilde{G}$ be a Gröbner base of $\tilde{I}$ with respect to a term order for eliminating u and v.

(iii) Set $G := \tilde{G} \cap D\langle t, \partial_t \rangle$.

(iv) Let P be an element of G. Then there exist $Q(s) \in D[s]$ and $\nu_1, \dots, \nu_p \in \mathbb{Z}$ such that

$$S_{1,\nu_1} \cdots S_{p,\nu_p} P = Q(-\partial_{t_1} t_1, \dots, -\partial_{t_p} t_p),$$

where $S_{j,\nu} = \partial_{t_j}^{\nu}$ if $\nu \geq 0$ and $S_{j,\nu} = t_j^{-\nu}$ otherwise. We denote this $Q(s)$ by $\psi(P)(s)$.

(v) Let I_1 be the ideal of $D[s]$ generated by $\{\psi(P)(s) \mid P \in G\}$.

Step 2 Compute $I_2 := (I_1 + D[s] \cdot F) \cap \mathbf{k}[x, s]$ with $F := f_1 \cdots f_p$ through a Gröbner base with respect to a term order for eliminating $\partial_{x_1}, \dots, \partial_{x_n}$.

Step 3 Compute the local Bernstein-Sato ideal at $a \in \mathbf{k}^n$ as follows:

(i) Compute a primary decomposition $I_2 = \Upsilon_1 \cap \cdots \cap \Upsilon_r$ in $\mathbf{k}[x, s]$.

(ii) Set $\sigma_a := \{i \in \{1, \dots, r\} \mid, a \in V(\Upsilon_i \cap \mathbf{k}[x])\}$.

(iii) If $\sigma_a = \emptyset$ then set $\mathcal{B}_{loc,a}(f) := \mathbf{k}[s]$. Otherwise set $\mathcal{B}_{loc,a}(f) := \bigcap_{i \in \sigma_a} \big(\Upsilon_i \cap \mathbf{k}[s]\big)$. This ideal intersection can be computed by Gröbner bases in $\mathbf{k}[x, s]$.

Output The local Bernstein-Sato ideal $\mathcal{B}_{loc,a}(f)$ of f at a. Step 3 also yields a stratification of $\mathbf{k}^n$ as is described in Corollary 5.

Note that Step 1 of this algorithm was given by Oaku and Takayama (1999); one can also use an alternative method introduced by Briançon and Maisonobe (2002). See also Ucha and Castro (2004), Gago-Vargas et al. (2005), Levandovskyy and Morales (2008). Steps 2 and 3 were introduced by (Oaku, 1997b, pp. 71–74) in the case $p = 1$.

As another consequence of our algorithm, we recover the following well-known fact:

Corollary 6. *Assume that* $\mathbf{k}$ *is algebraically closed. Then*

$$\mathcal{B}_{glob}(f) = \bigcap_{a \in \mathbf{k}^n} \mathcal{B}_{loc,a}(f).$$

Remark. With a slight generalisation of the construction and similar proofs one can obtain the following result when $\mathbf{k}$ is not supposed to be algebraically closed (see (Briançon and Maisonobe, 2002, Proposition 1.4)):

$$\mathcal{B}_{glob}(f) = \bigcap_{m \in \mathrm{SpecMax}(\mathbf{k}[x])} \mathcal{B}_{loc,m}(f),$$

where $\mathrm{SpecMax}(\mathbf{k}[x])$ is the set of the maximal ideals of $\mathbf{k}[x]$, and $\mathcal{B}_{loc,m}(f)$ is the set of $b(s) \in \mathbf{k}[s]$ such that $c(x) b(s) f^s \in D[s] F f^s$ with some $c(x) \in \mathbf{k}[x] \setminus m$.

The first statement in Proposition 3 says that the annihilators of f^s in $D[s]$ and in $D_a[s]$ have a common set of generators. Similarly

Proposition 7. *For $f \in \mathbf{k}[x]^p$ and $a \in \mathbf{k}^n$, $\mathrm{ann}_{\hat{D}_a[s]}(f^s)$ equals $\hat{D}_a[s] \cdot \mathrm{ann}_{D[s]}(f^s)$. If $f \in \mathbb{C}[x]^p$ and $a \in \mathbb{C}^n$, then $\mathrm{ann}_{\mathcal{D}_a[s]}(f^s)$ equals $\mathcal{D}_a[s] \cdot \mathrm{ann}_{D[s]}(f^s)$.*

Corollary 8. *If $f \in \mathbf{k}[x]^p$ and $a \in \mathbf{k}^n$, then $\mathcal{B}_{\mathbf{k}[x]_a}(f)$ coincides with $\mathcal{B}_{\mathbf{k}[[x-a]]}(f)$. If $f \in \mathbb{C}[x]^p$ and $a \in \mathbb{C}^n$, then $\mathcal{B}_{\mathbb{C}[x]_a}(f)$ coincides with $\mathcal{B}_{\mathcal{O}_a}(f)$.*

For this corollary, see also Proposition 1.7 in (Briançon and Maisonobe, 2002).

2. Proofs

All the proofs, except for Corollaries 5 and 6, concern a fixed point a. So we shall assume $a = 0$ in the sequel.

Proof of Lemma 1. Let us give the proof for $\mathrm{ann}_{\hat{D}_0\langle t,\partial_t\rangle}(f^s)$. The other cases are similar. Recall that $\mathrm{ann}_{\hat{D}_0\langle t,\partial_t\rangle}(f^s)$ is the left ideal $\{P \in \hat{D}_0\langle t, \partial_t\rangle \mid P \cdot f^s = 0\}$. Let P be in this ideal. Modulo the elements in (1), we may assume that $P \in \mathbf{k}[[x]][\partial_t]$. Let us write $P = \sum_\nu c_\nu \partial_t^\nu$ with $\nu \in \mathbb{N}^p$ and $\partial_t^\nu = \prod_{j=1}^p \partial_{t_j}^{\nu_j}$ and $c_\nu \in \mathbf{k}[[x]]$. Then

$$0 = Pf^s = \sum_\nu (-1)^{|\nu|} c_\nu \prod_{j=1}^{p} (s_j \cdots (s_j - \nu_j + 1) f_j^{-\nu_j}) f^s.$$

This equality takes place in the free module $\mathbf{k}[[x]][1/F, s] \cdot f^s$. Thus all the terms in the sum are zero, which implies that all the c_ν are zero. This completes the proof.

Proof of Proposition 2. Let us prove the second equality since the proof is the same for the first one. Let $b(s)$ be in $\mathbf{k}[s]$. If $b(s) \in \mathcal{B}_{loc,0}(f)$ then $b(s)f^s = P \cdot Ff^s$ for some $P \in D_0[s]$. Thus $b(s) - PF$ annihilates f^s, i.e. $b(s) \in (D_0[s]F + J_1) \cap \mathbf{k}[s] = J_3$. The converse implication can be proved in the same way.

Proof of Proposition 3. We have $I \subset J$ so $I_1 \subset J_1$ and then $D_0[s]I_1 \subset J_1$. Let us show the converse inclusion. Take P in $J_1 = (D_0\langle t, \partial_t\rangle \cdot I) \cap D_0[s]$. Writing P as an element in $D_0\langle t, \partial_t\rangle I$ and as an element of $D_0[s]$ we may clear the denominators and obtain the existence of $c(x) \in \mathbf{k}[x]$ with $c(0) \neq 0$ such that $c(x)P \in I \cap D[s]$. Thus P is in $D_0[s](I \cap D[s]) = D_0[s]I_1$. This ends the proof for the first equality. For the second one the arguments are exactly the same.

Proposition 4 is an obvious consequence of the following lemma:

Lemma 9.
(i) If $\Upsilon \subset \mathbf{k}[x, s]$ is an ideal with $0 \notin V(\Upsilon \cap \mathbf{k}[x])$ then
$(\mathbf{k}[x]_0[s] \cdot \Upsilon) \cap \mathbf{k}[s] = \mathbf{k}[s]$.
(ii) If $\Upsilon \subset \mathbf{k}[x, s]$ is a primary ideal with $0 \in V(\Upsilon \cap \mathbf{k}[x])$ then
$(\mathbf{k}[x]_0[s] \cdot \Upsilon) \cap \mathbf{k}[s] = \Upsilon \cap \mathbf{k}[s]$.
(iii) Given ideals $\Upsilon_1, \ldots, \Upsilon_r$ in $\mathbf{k}[x, s]$, we have:
$\mathbf{k}[x]_0[s] \cdot (\bigcap_{i=1}^r \Upsilon_i) = \bigcap_{i=1}^r (\mathbf{k}[x]_0[s] \cdot \Upsilon_i)$.

Proof. (i) If $0 \notin V(\Upsilon \cap \mathbf{k}[x])$, there exists $g \in \Upsilon \cap \mathbf{k}[x]$ such that $g(0) \neq 0$, which implies $1 = g^{-1}g \in \mathbf{k}[x]_0[s] \cdot \Upsilon$.

(ii) Let $f \in (\mathbf{k}[x]_0[s] \cdot \Upsilon) \cap \mathbf{k}[s]$. Then there exists $c \in \mathbf{k}[x]$ with $c(0) \neq 0$ such that $cf \in \Upsilon$. Assume, by contradiction, that $f \notin \Upsilon$. Then since Υ is primary, $c^l \in \Upsilon$ for some

$l \in \mathbb{N}$. This implies $c(0) = 0$, which is a contradiction. Thus $f \in \Upsilon \cap \mathbf{k}[s]$. This proves the left-right inclusion. The reverse one is trivial.

(iii) Since the left-right inclusion is trivial, let us prove the other one. Let f be in $\bigcap_{i=1}^{r}(\mathbf{k}[x]_0[s] \cdot \Upsilon_i)$. Then for each i, $c_i f \in \Upsilon_i$ for some $c_i \in \mathbf{k}[x]$ satisfying $c_i(0) \neq 0$. As a consequence, $(\prod_1^r c_i)f \in \bigcap_1^r \Upsilon_i$ and then $f \in \mathbf{k}[x]_0[s] \cdot (\bigcap_{i=1}^r \Upsilon_i)$. □

Now, let us work with arbitrary points $a \in \mathbf{k}^n$ and prove the two corollaries.

Proof of Corollary 5. First, it is clear that each W_σ is locally closed (or empty). Moreover, it is clear that any $a \in \mathbf{k}^n$ belongs to some W_σ (indeed, $a \in W_{\sigma_a}$ with the notations of Proposition 4 and Corollary 5). Thus we have a constructible stratification of $\mathbf{k}^n$. The constancy of the map $(a \mapsto \mathcal{B}_{loc,a}(f))$ on each W_σ follows from the obvious observation that if a and a' are two points in a W_σ then $\sigma_a = \sigma_{a'}$, which implies, in view of the whole algorithm and in particular Proposition 4, that $\mathcal{B}_{loc,a}(f) = \mathcal{B}_{loc,a'}(f)$.

Proof of Corollary 6. First, it is obvious from the definitions that $\mathcal{B}_{glob}(f)$ is included in any $\mathcal{B}_{loc,a}(f)$. So we have the inclusion $\mathcal{B}_{glob}(f) \subset \bigcap_a \mathcal{B}_{loc,a}(f)$. Let us prove the converse one. We follow the notations in Proposision 4 and Corollary 5. Let us fix $i \in \{1, \ldots, r\}$. Notice that since $\Upsilon_i \subset \mathbf{k}[x, s]$ is primary, $\Upsilon_i \cap \mathbf{k}[x]$ is also primary in $\mathbf{k}[x]$ and so $V(\Upsilon_i \cap \mathbf{k}[x])$ is irreducible. Set

$$\tau_i = \{k \in \{1, \ldots, r\} \mid V(\Upsilon_i \cap \mathbf{k}[x]) \subset V(\Upsilon_k \cap \mathbf{k}[x])\}.$$

Assume, by contradiction, that $W_{\tau_i} = \emptyset$. Then $V(\Upsilon_i \cap \mathbf{k}[x]) \subset \bigcup_{k \notin \tau_i} V(\Upsilon_k \cap \mathbf{k}[x])$ and by irreducibility of $V(\Upsilon_i \cap \mathbf{k}[x])$ it would be contained in $V(\Upsilon_k \cap \mathbf{k}[x])$ for some $k \notin \tau_i$, which is impossible. So let $a_i \in W_{\tau_i}$. Then we have $\mathcal{B}_{loc,a_i}(f) \subset \Upsilon_i \cap \mathbf{k}[s]$. As a consequence, we get

$$\bigcap_{a \in \mathbf{k}^n} \mathcal{B}_{loc,a}(f) \subset \bigcap_{i=1,\ldots,r} \mathcal{B}_{loc,a_i}(f) \subset \bigcap_{i=1,\ldots,r} (\Upsilon_i \cap \mathbf{k}[s]) = I_2 \cap \mathbf{k}[s] = \mathcal{B}_{glob}(f).$$

Proof of Proposition 7. We assume again that $a = 0$. We shall prove only the first statement. The arguments are the same for the second statement. First we have a natural isomorphism

$$\mathbf{k}[[x]] \otimes_{\mathbf{k}[x]} D[s]f^s \simeq \hat{D}_0[s] \otimes_{D[s]} D[s]f^s. \tag{2}$$

This gives a natural left $\hat{D}_0[s]$-module structure on the tensor product of the left-hand side.

Now let us start with the following exact sequence of $D[s]$-modules:

$$0 \to I_1 \to D[s] \to D[s]f^s \to 0.$$

By the flatness of $\hat{D}_0[s]$ over $D[s]$, we get an exact sequence of $\hat{D}_0[s]$-modules:

$$0 \to \hat{D}_0[s]I_1 \to \hat{D}_0[s] \to \hat{D}_0[s] \otimes D[s]f^s \to 0.$$

Thanks to the isomorphism (2), it remains to prove that $\mathbf{k}[[x]] \otimes_{\mathbf{k}[x]} D[s]f^s$ is naturally isomorphic to $\hat{D}_0[s]f^s$. We have an injective $D[s]$-morphism:

$$0 \to D[s]f^s \to \mathbf{k}[x][1/F, s]f^s.$$

Flatness of $\mathbf{k}[[x]]$ over $\mathbf{k}[x]$ implies the exactness of

$$0 \to \mathbf{k}[[x]] \otimes_{\mathbf{k}[x]} D[s]f^s \xrightarrow{\varphi} \mathbf{k}[[x]] \otimes_{\mathbf{k}[x]} \mathbf{k}[x][1/F, s]f^s.$$

On the other hand, there is a natural homomorphism

$$\mathbf{k}[[x]] \otimes_{\mathbf{k}[x]} \mathbf{k}[x][1/F, s]f^s \xrightarrow{\psi} \mathbf{k}[[x]][1/F, s]f^s.$$

An arbitrary element of $\mathbf{k}[[x]] \otimes_{\mathbf{k}[x]} \mathbf{k}[x][1/F, s]f^s$ is written in the form $\sum_\mu \hat{c}_\mu(x) \otimes s^\mu F^{-m} f^s$ with $\hat{c}_\mu(x) \in \mathbf{k}[[x]]$ and $m \in \mathbb{N}$, and it is sent to $\sum_\mu \hat{c}_\mu(x) s^\mu F^{-m} f^s$ by ψ. The latter is zero if and only if $\hat{c}_\mu(x) = 0$ for any μ. This shows that ψ is an isomorphism.

An arbitrary element $\sum_{\mu,\beta} \hat{c}_{\mu,\beta}(x) \otimes s^\mu \partial_x^\beta f^s$ of $\mathbf{k}[[x]] \otimes_{\mathbf{k}[x]} D[s]f^s$ with $\hat{c}_{\mu,\beta}(x) \in \mathbf{k}[[x]]$ is sent to $\sum_{\mu,\beta} \hat{c}_{\mu,\beta}(x) s^\mu \partial_x^\beta f^s$ by $\psi \circ \varphi$. This implies that the image of $\psi \circ \varphi$ coincides with $\hat{D}_0[s]f^s$. Thus $\psi \circ \varphi$ gives a natural isomorphism $\mathbf{k}[[x]] \otimes_{\mathbf{k}[x]} D[s]f^s \simeq \hat{D}_0[s]f^s$ and it is naturally $\hat{D}_0[s]$-linear. Hence we get an exact sequence of $\hat{D}_0[s]$-modules

$$0 \to \hat{D}_0[s]I_1 \to \hat{D}_0[s] \to \hat{D}_0[s]f^s \to 0$$

with natural maps. This completes the proof of Proposition 7.

Proof of Corollary 8. By Proposition 7 and the faithful flatness of $\hat{D}_0$ over D_0, we have

$$\begin{aligned} D_0[s] \cap \left(\mathrm{ann}_{\hat{D}_0[s]} f^s + \hat{D}_0[s]F\right) &= D_0[s] \cap \left(\hat{D}_0[s] \left(\mathrm{ann}_{D_0[s]} f^s + D_0[s]F\right)\right) \\ &= \mathrm{ann}_{D_0[s]} f^s + D_0[s]F. \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{B}_{\mathbf{k}[[x]]}(f) &= \mathbf{k}[s] \cap \left(\mathrm{ann}_{\hat{D}_0[s]} f^s + \hat{D}_0[s]F\right) \\ &= \mathbf{k}[s] \cap D_0[s] \cap \left(\mathrm{ann}_{\hat{D}_0[s]} f^s + \hat{D}_0[s]F\right) \\ &= \mathbf{k}[s] \cap \left(\mathrm{ann}_{D_0[s]} f^s + D_0[s]F\right) = B_{\mathbf{k}[x]_0}(f). \end{aligned}$$

The proof for $\mathcal{B}_{\mathcal{O}_0}(f)$ is the same.

3. Examples

Let us start with 'classical' results:

Lemma 10. *Assume $f \in \mathbf{k}[[x]]^p$.*

(i) $\mathcal{B}_{\mathbf{k}[[x]]}(u_1 f_1, \ldots, u_p f_p) = \mathcal{B}_{\mathbf{k}[[x]]}(f)$ *if* $u_1, \ldots, u_p$ *are units in* $\mathbf{k}[[x]]$.

(ii) $\mathcal{B}_{\mathbf{k}[[x]]}(f) = \mathbf{k}[s_1, \ldots, s_p] \cdot \mathcal{B}_{\mathbf{k}[[x]]}(f_1, \ldots, f_k)$ *if* $f_{k+1}, \ldots, f_p$ *are units in* $\mathbf{k}[[x]]$.

(iii) Let $\mathbf{K} \supset \mathbf{k}$ *be a field extension of* $\mathbf{k}$. *Then* $\mathcal{B}_{\mathbf{K}[[x]]}(f) = \mathbf{K}[s] \cdot \mathcal{B}_{\mathbf{k}[[x]]}(f)$, *and* $\mathcal{B}_{\mathbf{K}[x]}(f) = \mathbf{K}[s] \cdot \mathcal{B}_{\mathbf{k}[x]}(f)$ *if* $f \in \mathbf{k}[x]^p$.

(iv) Suppose $\mathbf{k} \subset \mathbb{C}$ *and* $f \in \mathbf{k}[x]^p$. *Let* $a \in \mathbf{k}^n$ *be such that* $f(a) = 0$ *and* f *is smooth at* a. *Then* $\mathcal{B}_{loc,a}(f)$ *is generated by* $\prod_{j=1}^p (s_j + 1)$.

Proof. (i) In the free module $\mathbf{k}[[x]][1/(u_1 \cdots u_p F), s] \cdot u^s f^s = \mathbf{k}[[x]][1/F, s] \cdot u^s f^s$, we have

$$\partial_{x_i} \cdot (u^s f^{s+1}) = \Big((\sum_{j=1}^p s_j \frac{\partial u_j}{\partial x_i} u_j^{-1} + \partial_{x_i}) \cdot f^{s+1} \Big) u^s,$$

where $u^s = u_1^{s_1} \cdots u_p^{s_p}$ and $f^{s+1} = F f_1^{s_1} \cdots f_p^{s_p}$. Thus by an easy induction one can prove that

$$\hat{D}_0[s] \cdot (u^s f^{s+1}) = (\hat{D}_0[s] \cdot f^{s+1}) u^s.$$

Set $f' = (u_1 f_1, \dots, u_p f_p)$. A polynomial $b(s) \in \mathbf{k}[s]$ belongs to $\mathcal{B}_{\mathbf{k}[[x]]}(f)$ if and only if there exists $P(s) \in \hat{D}_0[s]$ such that

$$b(s) f^s = P(s) \cdot f^{s+1}. \tag{3}$$

Multiplying $u^s = u_1^{s_1} \cdots u_p^{s_p}$, we get

$$\begin{aligned} b(s) {f'}^s &= (P(s) \cdot f^{s+1}) u^s = Q(s) \cdot (u^s f^{s+1}) \\ &= (Q(s) u_1^{-1} \cdots u_p^{-1}) {f'}_1^{s_1+1} \cdots {f'}_p^{s_p+1} \end{aligned}$$

with some $Q(s) \in \hat{D}_0[s]$. This implies the equality (i).

(ii) By using (i), we may assume that $f_{k+1} = \cdots = f_p = 1$. Moreover it suffices to assume $k = p-1$. Set $f' = (f_1, \dots, f_{p-1})$. The inclusion $\mathbf{k}[s]\mathcal{B}_{\mathbf{k}[[x]]}(f') \subset \mathcal{B}_{\mathbf{k}[[x]]}(f)$ is trivial. Let us prove the converse one. Set $s' = (s_1, \dots, s_{p-1})$ and let $b(s', s_p) \in \mathcal{B}_{\mathbf{k}[[x]]}(f)$. Then we have

$$b(s', s_p) {f'}^{s'} 1^{s_p} \in \hat{D}_0[s', s_p] f_1 \cdots f_{p-1} {f'}^{s'} 1^{s_p}.$$

Thus we see that $b(s', \lambda)$ belongs to $\mathcal{B}_{\mathbf{k}[[x]]}(f')$ for any $\lambda \in \mathbf{k}$. Let us write $b(s', s_p) = \sum_{l=0}^{d} c_k(s') s_p^l$. Let $\lambda_0, \dots, \lambda_d$ be pairwise distinct elements in $\mathbf{k}$. Then there exist $b_0(s')$, $\dots$, $b_d(s') \in \mathcal{B}_{\mathbf{k}[[x]]}(f')$ such that

$$\begin{pmatrix} 1 & \lambda_0 & \cdots & \lambda_0^d \\ \vdots & & & \vdots \\ 1 & \lambda_d & \cdots & \lambda_d^d \end{pmatrix} \begin{pmatrix} c_0(s') \\ \vdots \\ c_d(s') \end{pmatrix} = \begin{pmatrix} b_0(s') \\ \vdots \\ b_d(s') \end{pmatrix}.$$

This is an invertible Vandermonde matrix, from which we deduce that each c_l is in $\mathcal{B}_{\mathbf{k}[[x]]}(f')$. This implies $b(s', s_p) \in \mathbf{k}[s]\mathcal{B}_{\mathbf{k}[[x]]}(f')$.

(iii) The inclusion $\mathbf{K}[s] \cdot \mathcal{B}_{\mathbf{k}[[x]]}(f) \subset \mathcal{B}_{\mathbf{K}[[x]]}(f)$ is trivial. Let π be a $\mathbf{k}$-linear projection of $\mathbf{K}$ to $\mathbf{k}$ and let $b(s)$ belong to $\mathcal{B}_{\mathbf{K}[[x]]}(f)$. Then applying π to (3), we see that $\pi(b(s))$ belongs to $\mathcal{B}_{\mathbf{k}[[x]]}(f)$. Now fix an arbitrary term order for $\mathbf{K}[s]$. We may assume that the leading monomial of $\pi(b(s))$ coincides with that of $b(s)$. It follows that the set of the leading monomials of $\mathbf{K}[x] \cdot \mathcal{B}_{\mathbf{k}[[x]]}(f)$ contains the set of the leading monomials of $\mathcal{B}_{\mathbf{K}[[x]]}(f)$. Together with the above inclusion, this implies the equality of the two ideals.

(iv) From (iii) it follows that $\mathcal{B}_{loc,a}(f) = \mathcal{B}_{\mathbf{k}[x]_a}(f)$ equals $\mathcal{B}_{\mathbb{C}[x]_a}(f)$, which coincides with $\mathcal{B}_{\mathbb{C}\{x-a\}}(f)$ by Corollary 8. Thus (Briançon and Maynadier, 1999, Proposition 1.2) implies (iv). $\square$

The following examples were computed by using Risa/Asir (Noro el al.). This software is capable of computing Gröbner bases in the rings of polynomials and of differential operators as well as primary decompositions of polynomial ideals over the field $\mathbb{Q}$ of rational numbers. In the last paragraph of this section, we check that the results are also valid over $\mathbb{C}$. In the sequel, $\langle G \rangle$ denotes the ideal generated by the set G.

Example 1. This example is trivial in the sense that all the local Bernstein-Sato ideals can be computed by using Lemma 10. Let us define $f \in \mathbb{Q}[x, y]^3$ by

$$f = (f_1, f_2, f_3) = (x, y, 1 - x - y).$$

Only with Lemma 10 one can say that given $a \in \mathbb{C}^2$, $\mathcal{B}_{loc,a}(f)$ is equal to:

Table 1. Primary decomposition for Example 2

i	$\sqrt{\Upsilon_i}$	$\sqrt{\Upsilon_i \cap \mathbb{Q}[x,y]}$	$\Upsilon_i \cap \mathbb{Q}[s_1,s_2,s_3]$
1	$\langle s_1+1, y\rangle$	$\langle y\rangle$	$\langle s_1+1\rangle$
2	$\langle s_2+1, y-2x+1\rangle$	$\langle y-2x+1\rangle$	$\langle s_2+1\rangle$
3	$\langle s_3+1, y-x^2\rangle$	$\langle y-x^2\rangle$	$\langle s_3+1\rangle$
4	$\langle 2s_1+2s_3+3, x, y\rangle$	$\langle x, y\rangle$	$\langle 2s_1+2s_3+3\rangle$
5	$\langle 2s_2+2s_3+3, x-1, y-1\rangle$	$\langle x-1, y-1\rangle$	$\langle 2s_2+2s_3+3\rangle$
6	$\langle 2s_1+2s_3+5, x, y\rangle$	$\langle x, y\rangle$	$\langle 2s_1+2s_3+5\rangle$
7	$\langle 2s_2+2s_3+5, x-1, y-1\rangle$	$\langle x-1, y-1\rangle$	$\langle 2s_2+2s_3+5\rangle$

- $\mathbb{C}[s] = \mathbb{C}[s_1,s_2,s_3]$ for $a \notin \{x=0\} \cup \{y=0\} \cup \{x+y=1\}$,
- $\langle (s_1+1)\rangle$ for $a \in \{x=0\} \setminus \{(0,0),(0,1)\}$,
- $\langle (s_2+1)\rangle$ for $a \in \{y=0\} \setminus \{(0,0),(1,0)\}$,
- $\langle (s_3+1)\rangle$ for $a \in \{x+y=1\} \setminus \{(0,1),(1,0)\}$,
- $\langle (s_1+1)(s_2+1)\rangle$ if $a=(0,0)$, $\langle (s_1+1)(s_3+1)\rangle$ if $a=(0,1)$, $\langle (s_2+1)(s_3+1)\rangle$ if $a=(1,0)$.

By using Corollary 6 one has

$$\mathcal{B}_{glob}(f) = \langle (s_1+1)(s_2+1)(s_3+1)\rangle.$$

We notice that the global Bernstein-Sato ideal is different from all the local ones. On the other hand, we find the following primary decomposition for $I_2 \subset \mathbb{Q}[x,y,s_1,s_2,s_3]$ in Algorithm 1:

$$I_2 = \Upsilon_1 \cap \Upsilon_2 \cap \Upsilon_3 \text{ with } \Upsilon_j = \langle s_j+1, f_j\rangle,$$

which obviously recovers the results above.

Example 2. Define $f=(f_1,f_2,f_3) \in \mathbb{Q}[x,y]^3$ by $f=(y, y-2x+1, y-x^2)$. The computed primary decomposition of the ideal I_2 in Algorithm 1 has seven primary components Υ_i. For each of them, we present its radical, the radical of the intersection with $\mathbb{Q}[x,y]$, and the intersection with $\mathbb{Q}[s_1,s_2,s_3]$ in Table 1.

From these data, we can read off the local Bernstein-Sato ideal at each point of $\mathbb{Q}^2$. For example, at $(0,0)$ we have

$$\begin{aligned}\mathcal{B}_{loc,0}(f) &= (\Upsilon_1 \cap \Upsilon_3 \cap \Upsilon_4 \cap \Upsilon_6) \cap \mathbb{Q}[s_1,s_2,s_3]\\ &= \langle (s_1+1)(s_3+1)(2s_1+2s_3+3)(2s_1+2s_3+5)\rangle.\end{aligned}$$

The global Bernstein-Sato ideal is

$$\mathcal{B}_{glob}(f) = \langle (s_1+1)(s_2+1)(s_3+1)(2s_1+2s_3+3)(2s_1+2s_3+5)(2s_2+2s_3+3)(2s_2+2s_3+5)\rangle,$$

which is different from all the local Bernstein-Sato ideals.

Example 3. Here $f \in \mathbb{Q}[x,y,z]^2$ is given by $(f_1,f_2) = (z, x^4+y^4+2zx^2y^2)$. This important example is taken from (Briançon and Maynadier, 1999), where it is proved that $\mathcal{B}_{loc,0}(f)$ is not principal. However, its generators have not been given explicitly

Table 2. Primary decomposition for Example 3

i	$\sqrt{\Upsilon_i}$	$\sqrt{\Upsilon_i \cap \mathbb{Q}[x,y,z]}$	$\Upsilon_i \cap \mathbb{Q}[s_1, s_2]$
1	$\langle z, s_1+1\rangle$	$\langle z\rangle$	$\langle s_1+1\rangle$
2	$\langle f_2, s_2+1\rangle$	$\langle f_2\rangle$	$\langle s_2+1\rangle$
3	$\langle x, y, s_2+1\rangle$	$\langle x, y\rangle$	$\langle (s_2+1)^2\rangle$
4	$\langle x, y, 2s_2+1\rangle$	$\langle x, y\rangle$	$\langle 2s_2+1\rangle$
5	$\langle x, y, 4s_2+3\rangle$	$\langle x, y\rangle$	$\langle 4s_2+3\rangle$
6	$\langle x, y, 4s_2+5\rangle$	$\langle x, y\rangle$	$\langle 4s_2+5\rangle$
7	$\langle x, y, z, s_1+2, 2s_2+3\rangle$	$\langle x, y, z\rangle$	$\langle s_1+2, 2s_2+3\rangle$
8	$\langle x, y, z-1, 2s_2+3\rangle$	$\langle x, y, z-1\rangle$	$\langle 2s_2+3\rangle$
9	$\langle x, y, z+1, 2s_2+3\rangle$	$\langle x, y, z+1\rangle$	$\langle 2s_2+3\rangle$

as far as the present authors know. The ideal $I_2 \subset \mathbb{Q}[x, y, z, s_1, s_2]$ has nine primary components Υ_i (Table 2).

As a consequence, the local Bernstein-Sato ideal $\mathcal{B}_{loc,0}(f)$ is generated by two elements:

$$\begin{aligned}\mathcal{B}_{loc,0}(f) = \langle &(s_1+1)(s_2+1)^2(2s_2+1)(4s_2+3)(4s_2+5)(s_1+2),\\ &(s_1+1)(s_2+1)^2(2s_2+1)(4s_2+3)(4s_2+5)(2s_2+3)\rangle,\end{aligned}$$

while the global Bernstein-Sato ideal $\mathcal{B}_{glob}(f)$ is principal:

$$\mathcal{B}_{glob}(f) = \langle (s_1+1)(s_2+1)^2(2s_2+1)(2s_2+3)(4s_2+3)(4s_2+5)\rangle.$$

Example 4. Let us consider $f = (f_1, f_2) = (z, x^5 + y^5 + zx^2y^3) \in \mathbb{Q}[x, y, z]^2$. The computed primary decomposition of I_2 consists of twelve terms Υ_i (Table 3).

We conclude that $\mathcal{B}_{glob}(f)$ and $\mathcal{B}_{loc,0}(f)$ are equal and generated by the following three elements:

- $(s_1+1)(s_2+1)^2(5s_2+2)(5s_2+3)(5s_2+4)(5s_2+6)(s_1+2)(s_1+3)(s_1+4)(s_1+5)$,
- $(s_1+1)(s_2+1)^2(5s_2+2)(5s_2+3)(5s_2+4)(5s_2+6)(5s_2+7)(s_1+2)$,
- $(s_1+1)(s_2+1)^2(5s_2+2)(5s_2+3)(5s_2+4)(5s_2+6)(5s_2+7)(5s_2+8)$.

Example 5. Here $f \in \mathbb{Q}[x, y, z]^2$ is given by $(f_1, f_2) = (xz, x^4 + y^4 + 2zx^2y^2)$. The computed primary decomposition of I_2 consists of 12 components Υ_i (Table 4).

The local Bernstein-Sato ideal $\mathcal{B}_{loc,0}(f)$ at $(0, 0, 0)$ coincides with the global Bernstein-Sato ideal $\mathcal{B}_{glob}(f)$, which is generated by the following two elements:

- $(s_1+1)^2(s_2+1)(s_1+4s_2+2)(s_1+4s_2+3)(s_1+4s_2+4)$
  $(s_1+4s_2+5)(s_1+4s_2+6)(s_1+4s_2+7)(s_1+2)$,
- $(s_1+1)^2(s_2+1)(s_1+4s_2+2)(s_1+4s_2+3)(s_1+4s_2+4)$
  $(s_1+4s_2+5)(s_1+4s_2+6)(s_1+4s_2+7)(2s_2+3)$.

Table 3. Primary decomposition for Example 4

i	$\sqrt{\Upsilon_i}$	$\sqrt{\Upsilon_i \cap \mathbb{Q}[x,y,z]}$	$\Upsilon_i \cap \mathbb{Q}[s_1, s_2]$
1	$\langle z, s_1+1 \rangle$	$\langle z \rangle$	$\langle s_1+1 \rangle$
2	$\langle f_2, s_2+1 \rangle$	$\langle f_2 \rangle$	$\langle s_2+1 \rangle$
3	$\langle x, y, s_2+1 \rangle$	$\langle x, y \rangle$	$\langle (s_2+1)^2 \rangle$
4	$\langle x, y, 5s_2+2 \rangle$	$\langle x, y \rangle$	$\langle 5s_2+2 \rangle$
5	$\langle x, y, 5s_2+3 \rangle$	$\langle x, y \rangle$	$\langle 5s_2+3 \rangle$
6	$\langle x, y, 5s_2+4 \rangle$	$\langle x, y \rangle$	$\langle 5s_2+4 \rangle$
7	$\langle x, y, 5s_2+6 \rangle$	$\langle x, y \rangle$	$\langle 5s_2+6 \rangle$
8	$\langle x, y, z, s_1+2, 5s_2+7 \rangle$	$\langle x, y, z \rangle$	$\langle s_1+2, 5s_2+7 \rangle$
9	$\langle x, y, z, s_1+3, 5s_2+7 \rangle$	$\langle x, y, z \rangle$	$\langle s_1+3, 5s_2+7 \rangle$
10	$\langle x, y, z, s_1+4, 5s_2+7 \rangle$	$\langle x, y, z \rangle$	$\langle s_1+4, 5s_2+7 \rangle$
11	$\langle x, y, z, s_1+5, 5s_2+7 \rangle$	$\langle x, y, z \rangle$	$\langle s_1+5, 5s_2+7 \rangle$
12	$\langle x, y, z, s_1+2, 5s_2+8 \rangle$	$\langle x, y, z \rangle$	$\langle s_1+2, 5s_2+8 \rangle$

Table 4. Primary decomposition for Example 5

i	$\sqrt{\Upsilon_i}$	$\sqrt{\Upsilon_i \cap \mathbb{Q}[x,y,z]}$	$\Upsilon_i \cap \mathbb{Q}[s_1, s_2]$
1	$\langle s_2+1, f_2 \rangle$	$\langle f_2 \rangle$	$\langle s_2+1 \rangle$
2	$\langle s_1+1, z \rangle$	$\langle z \rangle$	$\langle s_1+1 \rangle$
3	$\langle s_1+1, x \rangle$	$\langle x \rangle$	$\langle s_1+1 \rangle$
4	$\langle 2s_2+3, s_1+2, x, y, z \rangle$	$\langle x, y, z \rangle$	$\langle s_1+2, 2s_2+3 \rangle$
5	$\langle s_1+4s_2+2, x, y \rangle$	$\langle x, y \rangle$	$\langle s_1+4s_2+2 \rangle$
6	$\langle s_1+4s_2+3, x, y \rangle$	$\langle x, y \rangle$	$\langle s_1+4s_2+3 \rangle$
7	$\langle s_1+4s_2+4, x, y \rangle$	$\langle x, y \rangle$	$\langle s_1+4s_2+4 \rangle$
8	$\langle s_1+4s_2+5, x, y \rangle$	$\langle x, y \rangle$	$\langle s_1+4s_2+5 \rangle$
9	$\langle s_1+4s_2+6, x, y \rangle$	$\langle x, y \rangle$	$\langle s_1+4s_2+6 \rangle$
10	$\langle s_1+4s_2+7, x, y \rangle$	$\langle x, y \rangle$	$\langle s_1+4s_2+7 \rangle$
11	$\langle s_1+1, x, z \rangle$	$\langle x, z \rangle$	$\langle (s_1+1)^2 \rangle$
12	$\langle 4s_2+5, s_1+2, x, y, z \rangle$	$\langle x, y, z \rangle$	$\langle s_1+2, 4s_2+5 \rangle$

Example 6. Set $f = (f_1, f_2) = (z^2, x^6 + y^6 + 2zx^3y^3) \in \mathbb{Q}[x,y,z]^2$. The computed primary decomposition of I_2 consists of 15 components Υ_i (Table 5).

The local Bernstein-Sato ideal $\mathcal{B}_{loc,0}(f)$ at $(0,0,0)$ is generated by the following two

Table 5. Primary decomposition for Example 6

i	$\sqrt{\Upsilon_i}$	$\sqrt{\Upsilon_i \cap \mathbb{Q}[x,y,z]}$	$\Upsilon_i \cap \mathbb{Q}[s_1,s_2]$
1	$\langle s_2+1, f_2\rangle$	$\langle f_2\rangle$	$\langle s_2+1\rangle$
2	$\langle 2s_2+1, x, y\rangle$	$\langle x, y\rangle$	$\langle 2s_2+1\rangle$
3	$\langle 2s_2+3, 2s_1+3, x, y, z\rangle$	$\langle x, y, z\rangle$	$\langle 2s_1+3, 2s_2+3\rangle$
4	$\langle 2s_2+3, x, y, z-1\rangle$	$\langle x, y, z-1\rangle$	$\langle 2s_2+3\rangle$
5	$\langle 2s_2+3, x, y, z+1\rangle$	$\langle x, y, z+1\rangle$	$\langle 2s_2+3\rangle$
6	$\langle 3s_2+1, x, y\rangle$	$\langle x, y\rangle$	$\langle 3s_2+1\rangle$
7	$\langle 3s_2+2, x, y\rangle$	$\langle x, y\rangle$	$\langle 3s_2+2\rangle$
8	$\langle 3s_2+4, x, y\rangle$	$\langle x, y\rangle$	$\langle 3s_2+4\rangle$
9	$\langle 6s_2+5, x, y\rangle$	$\langle x, y\rangle$	$\langle 6s_2+5\rangle$
10	$\langle 6s_2+7, x, y\rangle$	$\langle x, y\rangle$	$\langle 6s_2+7\rangle$
11	$\langle 3s_2+5, 2s_1+3, x, y, z\rangle$	$\langle x, y, z\rangle$	$\langle 2s_1+3, 3s_2+5\rangle$
12	$\langle s_1+1, z\rangle$	$\langle z\rangle$	$\langle s_1+1\rangle$
13	$\langle 2s_1+1, z\rangle$	$\langle z\rangle$	$\langle 2s_1+1\rangle$
14	$\langle s_2+1, x, y\rangle$	$\langle x, y\rangle$	$\langle (s_2+1)^2\rangle$
15	$\langle 3s_2+4, 2s_1+3, x, y, z\rangle$	$\langle x, y, z\rangle$	$\langle 2s_1+3, 3s_2+4\rangle$

elements:

- $(s_1+1)(2s_1+1)(s_2+1)^2(2s_2+1)(3s_2+1)(3s_2+2)(3s_2+4)(6s_2+5)(6s_2+7)$ $(2s_2+3)(3s_2+5)$,
- $(s_1+1)(2s_1+1)(s_2+1)^2(2s_2+1)(3s_2+1)(3s_2+2)(3s_2+4)(6s_2+5)(6s_2+7)$ $(2s_1+3)$.

The global Bernstein-Sato ideal $\mathcal{B}_{glob}(f)$ is different from $\mathcal{B}_{loc,0}(f)$ and is generated by the following two elements:

- $(s_1+1)(2s_1+1)(s_2+1)^2(2s_2+1)(3s_2+1)(3s_2+2)(3s_2+4)(6s_2+5)(6s_2+7)$ $(2s_2+3)(3s_2+5)$,
- $(s_1+1)(2s_1+1)(s_2+1)^2(2s_2+1)(3s_2+1)(3s_2+2)(3s_2+4)(6s_2+5)(6s_2+7)$ $(2s_1+3)(2s_2+3)$.

3.1. Validity of the computations over $\mathbb{C}$

First, let us state some general results useful in the sequel. We give proofs to these results for the sake of completeness although they might be well-known to specialists. Let $\mathbf{k} \subseteq \mathbf{K}$ be two fields of characteristic zero.

Lemma 11. *Let J be an ideal in $\mathbf{k}[y,z] = \mathbf{k}[y_1,\ldots,y_q,z_1,\ldots,z_r]$. Then*

$$(\mathbf{K}[y,z]\cdot J) \cap \mathbf{K}[y] = \mathbf{K}[y]\cdot(J \cap \mathbf{k}[y]).$$

Proof. Let us consider $g \in (\mathbf{K}[y,z] \cdot J) \cap \mathbf{K}[y]$. Let $f_1, \ldots, f_r$ be generators of J and let us write $g = \sum_j u_j f_j$ with $u_j \in \mathbf{K}[y,z]$. Let e_l ($l \in L$) be a basis of the $\mathbf{k}$-vector space generated by all the coefficients of the u_j's. Then one can write $g \in \oplus_l J e_l \subset \oplus_l \mathbf{k}[x,y] e_l$. Since $g \in \mathbf{K}[y]$, we obtain $g \in \oplus_l (J \cap \mathbf{k}[y]) e_l$. The left-right inclusion is proven. The other one being trivial, the proof is complete. □

The following lemma can be proved in the same way:

Lemma 12. *Let J be an ideal in $\mathbf{k}[x] = \mathbf{k}[x_1, \ldots, x_n]$.*
(i) $(\mathbf{K}[x]J) \cap \mathbf{k}[x] = J$.
(ii) If $\mathbf{K}[x]J$ is primary in $\mathbf{K}[x]$ then J is primary in $\mathbf{k}[x]$.

We shall denote by $\mathrm{Gal}(\mathbf{K}/\mathbf{k})$ the Galois group of the extension $\mathbf{K}/\mathbf{k}$. If $\tau \in \mathrm{Gal}(\mathbf{K}/\mathbf{k})$ then we shall also denote by τ the induced ring automorphism of $\mathbf{K}[x] = \mathbf{K}[x_1, \ldots, x_n]$.

Lemma 13. *Assume that $\mathbf{K}/\mathbf{k}$ is a Galois extension. Let $J \subset \mathbf{K}[x] = \mathbf{K}[x_1, \ldots, x_n]$ be an ideal. Suppose $\tau(J) \subset J$ for any $\tau \in \mathrm{Gal}(\mathbf{K}/\mathbf{k})$. Then there exists an ideal $J_0 \subset \mathbf{k}[x]$ such that $J = \mathbf{K}[x]J_0$.*

Proof. Given $\tau \in \mathrm{Gal}(\mathbf{K}/\mathbf{k})$, we have $\tau^{-1}(J) \subset J$, from which we deduce that $\tau(J) = J$.

Let G be the reduced Gröbner basis of J with respect to a fixed term order. In view of Buchberger's criterion, we see that $\tau(G)$ is also the reduced Gröbner basis of J for any $\tau \in \mathrm{Gal}(\mathbf{K}/\mathbf{k})$. Therefore, $\tau(g) = g$ holds for any $g \in G$. Thus, τ fixes each coefficient of g. Since τ is arbitrary and the extension $\mathbf{K}/\mathbf{k}$ is Galois, we get $g \in \mathbf{k}[x]$. □

Proposition 14. *Let Υ be a primary ideal of $\mathbf{k}[x] = \mathbf{k}[x_1, \ldots, x_n]$ and $\mathbf{K}$ be a field containing the algebraic closure of $\mathbf{k}$. If the radical of $\mathbf{K}[x]\Upsilon$ is a prime ideal of $\mathbf{K}[x]$, then $\mathbf{K}[x]\Upsilon$ is a primary ideal of $\mathbf{K}[x]$.*

Proof. Take an irredundant primary decomposition

$$\mathbf{K}[x]\Upsilon = \bigcap_{j=0}^{m} \Upsilon_j^{\mathbf{K}} \tag{4}$$

in $\mathbf{K}[x]$. We assume by contradiction that $m \geq 1$.

The algorithms for computing a primary decomposition imply that there exists a finite Galois extension $\mathbf{K}'$ of $\mathbf{k}$ such that $\Upsilon_j^{\mathbf{K}}$ are defined over $\mathbf{K}'$ (see e.g. (Greuel and Pfister, 2002, Chapter 4)). Then the field extension from $\mathbf{K}'$ to $\mathbf{K}$ is trivial in the sense that the primarity of each component is preserved. Thus we may now assume $\mathbf{K}' = \mathbf{K}$.

Since the radicals $\sqrt{\Upsilon_j^{\mathbf{K}}}$ are distinct and $\sqrt{\mathbf{K}[x]\Upsilon}$ is prime, we may assume, without loss of generality, that $\sqrt{\Upsilon_0^{\mathbf{K}}} = \sqrt{\mathbf{K}[x]\Upsilon}$ and the dimension of $\sqrt{\Upsilon_j^{\mathbf{K}}}$ is less than that of $\sqrt{\mathbf{K}[x]\Upsilon}$ for $j = 1, \ldots, m$.

Let τ be an element of the Galois group $\mathrm{Gal}(\mathbf{K}/\mathbf{k})$. Then

$$\mathbf{K}[x]\Upsilon = \bigcap_{j=0}^{m} \tau(\Upsilon_j^{\mathbf{K}}) \tag{5}$$

is also an irredundant primary decomposition. Since the non-embedded primary components are unique, we have $\tau(\Upsilon_0^{\mathbf{K}}) = \Upsilon_0^{\mathbf{K}}$. Since $\tau \in \mathrm{Gal}(\mathbf{K}/\mathbf{k})$ is arbitrary, this implies, by Lemma 13, that $\Upsilon_0^{\mathbf{K}}$ is defined over $\mathbf{k}$, i.e., there is an ideal Υ_0 of $\mathbf{k}[x]$ such that $\Upsilon_0^{\mathbf{K}} = \mathbf{K}[x]\Upsilon_0$. By Lemma 12 (2), Υ_0 is primary in $\mathbf{k}[x]$.

For each $j = 1, \ldots, m$, by Lemma 13, there exists an ideal Υ_j of $\mathbf{k}[x]$ such that

$$\mathbf{K}[x]\Upsilon_j = \bigcap_{\tau \in \mathrm{Gal}(\mathbf{K}/\mathbf{k})} \tau(\Upsilon_j^{\mathbf{K}}). \tag{6}$$

Moreover, Υ_j is primary in $\mathbf{k}[x]$. Indeed, assume $f, g \in \mathbf{k}[x]$ satisfy $fg \in \Upsilon_j$ and $f \notin \Upsilon_j$. Then there exists $\tau \in \mathrm{Gal}(\mathbf{K}/\mathbf{k})$ such that $f \notin \tau(\Upsilon_j^{\mathbf{K}})$. Since f is fixed by every element of $\mathrm{Gal}(\mathbf{K}/\mathbf{k})$, it follows that $f \notin \tau(\Upsilon_j^{\mathbf{K}})$ for any $\tau \in \mathrm{Gal}(\mathbf{K}/\mathbf{k})$. In particular, $f \notin \Upsilon_j^{\mathbf{K}}$. Hence there exists an integer ν such that $g^\nu \in \Upsilon_j^{\mathbf{K}}$. Since $g^\nu = \tau(g^\nu) \in \tau(\Upsilon_j^{\mathbf{K}})$ for any $\tau \in \mathrm{Gal}(\mathbf{K}/\mathbf{k})$, we have $g^\nu \in \mathbf{K}[x]\Upsilon_j$, which implies, by Lemma 12 (1), $g^\nu \in \Upsilon_j$.

Combining equalities (6) and (5), we get

$$\mathbf{K}[x]\Upsilon = \bigcap_{j=0}^{m} (\mathbf{K}[x]\Upsilon_j).$$

Using Lemma 12 (1), we obtain a (not necessarily irredundant) primary decomposition in $\mathbf{k}[x]$:

$$\Upsilon = \bigcap_{j=0}^{m} \Upsilon_j.$$

Since Υ is primary and $\dim(\Upsilon) = \dim(\Upsilon_0) > \dim(\Upsilon_j)$ for $j \geq 1$, the uniqueness of the number of components in irredundant primary decomposition implies that $\Upsilon_0 \subset \Upsilon_j$ for $j = 1, \ldots, m$. Hence we get

$$\Upsilon_0^{\mathbf{K}} = \mathbf{K}[x]\Upsilon_0 \subset \mathbf{K}[x]\Upsilon_j = \bigcap_{\tau \in \mathrm{Gal}(\mathbf{K}/\mathbf{k})} \tau(\Upsilon_j^{\mathbf{K}}) \subset \Upsilon_j^{\mathbf{K}}$$

for $j = 1, \ldots, m$. This contradicts the irredundancy of (4). □

Let us return to the validity proof of the examples over $\mathbb{C}$. For a given $f \in \mathbb{Q}[x]^p$ with $x = (x_1, \ldots, x_n)$, we compute the ideals I, I_1, I_2 introduced after Lemma 1 which are defined over $\mathbb{Q}$, i.e., with $\mathbf{k} = \mathbb{Q}$. We denote by $I^{\mathbb{C}}$, $I_1^{\mathbb{C}}$ and $I_2^{\mathbb{C}}$ the ideals defined over $\mathbb{C}$ obtained (theoretically) by the same processes but with $\mathbf{k} = \mathbb{C}$. Then it is easy to see that

$$I^{\mathbb{C}} = D_{\mathbb{C}[x]}\langle t, \partial_t \rangle \cdot I, \quad I_1^{\mathbb{C}} = D_{\mathbb{C}[x]}[s] \cdot I_1, \quad I_2^{\mathbb{C}} = \mathbb{C}[x, s] \cdot I_2.$$

Thus the construction is the same over $\mathbb{Q}$ and over $\mathbb{C}$ up to Step 2 of Algorithm 1.

Claim. Let $I_2 = \Upsilon_1 \cap \cdots \cap \Upsilon_r$ be a primary decomposition of I_2 in $\mathbb{Q}[x, s]$ for f of Examples 1–6. Then

$$\mathbb{C}[x, s]I_2 = (\mathbb{C}[x, s]\Upsilon_1) \cap \cdots \cap (\mathbb{C}[x, s]\Upsilon_r)$$

is also a primary decomposition in $\mathbb{C}[x, s]$.

Proof. By Proposition 14, it suffices to prove that each $\mathbb{C}[x, s]\sqrt{\Upsilon_i}$ is prime in $\mathbb{C}[x, s]$. In all the examples, $\sqrt{\Upsilon_i}$ is either generated by first degree polynomials, or of the form

Table 6. Timing data (in seconds)

Input	Step 1	Step 2	Step 3	Total time	Step 2 (HE)
Example 2	0.1	8.6	0.8	9.5	0.5
Example 3	0.1	1.3	0.8	2.2	0.6
Example 4	1.0	211.2	4.2	216.4	7.8
Example 5	0.2	68.8	7.6	76.6	46.1
Example 6	2.3	343.4	23.9	369.6	114.6
Example 7	0.5	17.0	0.8	18.3	0.5
Example 8	0.1	45.4	1.6	47.1	1.7
Example 9	0.9	–	34.0	(171.7)	136.8

$\langle s_j + 1, f_j \rangle$ with a polynomial f_j of degree greater than one ($j = 3$ for Example 2 and $j = 2$ for Examples 3–6). In the first case, $\mathbb{C}[x, s]\sqrt{\Upsilon_i}$ is obviously prime. It remains to analyse the second case.

We have a ring isomorphism $\mathbb{C}[x, s]/\langle s_1 + 1, f_j \rangle \simeq \mathbb{C}[x, s_2, \ldots, s_p]/\langle f_j \rangle$. In view of this relation, it is enough to prove that each f_j is irreducible over $\mathbb{C}$. For Example 2, $f_3 = y - x^2$ is obviously irreducible over $\mathbb{C}$. For Examples 3–6, each f_2 has a form $f_2 = u(x, y) + v(x, y)z$ with polynomials $u(x, y)$ and $v(x, y)$ in x, y. Since f_2 is first order with respect to z, it is reducible over $\mathbb{C}$ if and only if $u(x, y)$ and $v(x, y)$ have a non-constant common factor in $\mathbb{C}[x, y]$, which is obviously not the case since each $v(x, y)$ is a monomial. □

Now that the claim is proved, applying Proposition 4 and Lemma 11 we get, for any $\alpha \in \mathbb{C}^n$,

$$\mathcal{B}_{loc,\alpha}(f) = \bigcap_{i \in \sigma_\alpha} ((\mathbb{C}[x, s]\Upsilon_i) \cap \mathbb{C}[s]) = \mathbb{C}[s] \cdot (\bigcap_{i \in \sigma_\alpha} (\Upsilon_i \cap \mathbb{Q}[s])),$$

where $\sigma_\alpha = \{i \mid 1 \leq i \leq r, \alpha \in V((\mathbb{C}[x, s]\Upsilon_i) \cap \mathbb{C}[x])\}$. Notice that by Lemma 11, $(\mathbb{C}[x, s]\Upsilon_i) \cap \mathbb{C}[x] = \mathbb{C}[x](\Upsilon_i \cap \mathbb{Q}[x])$. Consequently the computations of Examples 1–6 done over $\mathbb{Q}$ are also valid over $\mathbb{C}$. That is, they provide us with stratifications over $\mathbb{C}$ such that the local Bernstein-Sato ideal remains the same on each strata.

4. Some remarks on the implementation

Our implementation is realized as a library file “bsi” of Risa/Asir, which will be contained in its distribution (Noro el al.) and/or will be put on the website of the second named author. Table 6 shows the running time (in seconds) of each step of Algorithm 1 on 2.2GHz Intel Core 2 Duo processor with 2GB RAM. In the table, Example 7 is $f = (x^3 + y^2, x^2 + y^3)$, Example 8 is $f = (x, x + y^2, x + z^2)$, and Example 9 is $f = (xyz, x^3 + y^3 + z^3)$. The local Bernstein-Sato ideals for Examples 7, 8, 9 at the origin are principal.

At least for these examples, Step 2 is the most time-consuming part, where the elimination is done in one step. Eliminating variables one by one in a suitable order often speeds up the computation of Step 2 as is shown in the right-most column (HE for heuristic elimination) of the table. However, it would be difficult to predict the fastest strategy in advance. Step 1 should be improved by adopting the alternative method of Briançon and Maisonobe (2002), as was suggested by Ucha and Castro (2004) and Gago-Vargas et al. (2005). This would require computations in a ring of differential-difference operators, which are not yet available with Risa/Asir.

At the time of this writing, the authors do not know any other systems which are capable of computing local Bernstein-Sato ideals. However, a computer algebra system Singular (Greuel and Pfister (2002)) provides a package "dmod.lib" (Levandovskyy and Morales (2008)) for computing global Bernstein-Sato ideals by the method of Briançon and Maisonobe (2002). In our experiments, the performance of our implementation for global Bernstein-Sato ideals is comparable to that of "dmod.lib".

Acknowledgements

We are grateful to Masayuki Noro for the helpful suggestions on Gröbner base computations with Risa/Asir in the ring of differential operators. We would also like to thank the anonymous referees for the constructive comments on the improvement of the present paper.

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