Completely distinguishable projections of spatial graphs*

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1. Completely distinguishable projection

Let G be a finite graph. We give a label to each of vertices and edges of G and denote the set of all vertices and the set of all edges of a graph G by $V(G)$ and $E(G)$, respectively. An embedding of G into $\mathbf{R}^3$ is called a *spatial embedding* of G or simply a *spatial graph*. Two spatial embeddings f and g of G are said to be *ambient isotopic* if there exists an orientation preserving homeomorphism $\Phi : \mathbf{R}^3 \to \mathbf{R}^3$ such that $\Phi \circ f = g$. A graph G is said to be *planar* if there exists an embedding of G into $\mathbf{R}^2$. A spatial embedding of a planar graph G is said to be *trivial* if it is ambient isotopic to an embedding of G into $\mathbf{R}^2 \subset \mathbf{R}^3$. Such an embedding is unique up to ambient isotopy in $\mathbf{R}^3$ [7].

A *regular projection* of G is an immersion $G \to \mathbf{R}^2$ whose multiple points are only finitely many transversal double points away from vertices. For a regular projection $\hat{f}$ of G with p double points, we can obtain 2^p regular diagrams of the spatial embeddings of G from $\hat{f}$ by giving over/under information to each double point. Then we say that a spatial embedding f of G is *obtained from* $\hat{f}$ if f is ambient isotopic to a spatial embedding of G which is represented by one of these 2^p regular diagrams. If these 2^p regular diagrams represent mutually different spatial embeddings of G up to ambient isotopy, then we say that $\hat{f}$ is *completely distinguishable*. A completely distinguishable projection of G is said to be *trivial* if it has no double points. Therefore every planar graph has a trivial completely

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distinguishable projection.

Example 1.1. Let G be the octahedron graph and $\hat{f}$ a regular projection of G as illustrated in Fig. 1.1. We can obtain the eight spatial embeddings $g_1, g_2, \ldots, g_8$ of G from $\hat{f}$ as illustrated in Fig. 1.2. Then, by observing the constituent knots and links we can see that g_i and g_j are not ambient isotopic for $i \neq j$. Thus $\hat{f}$ is a non-trivial completely distinguishable projection.

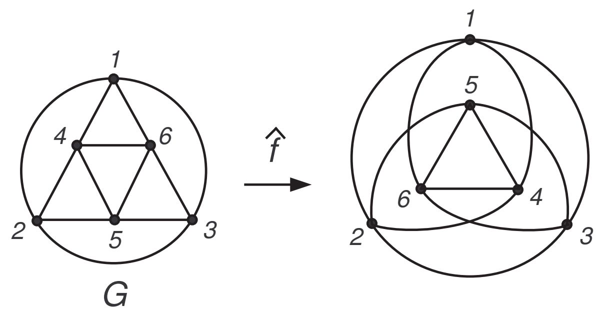

Fig. 1.1.

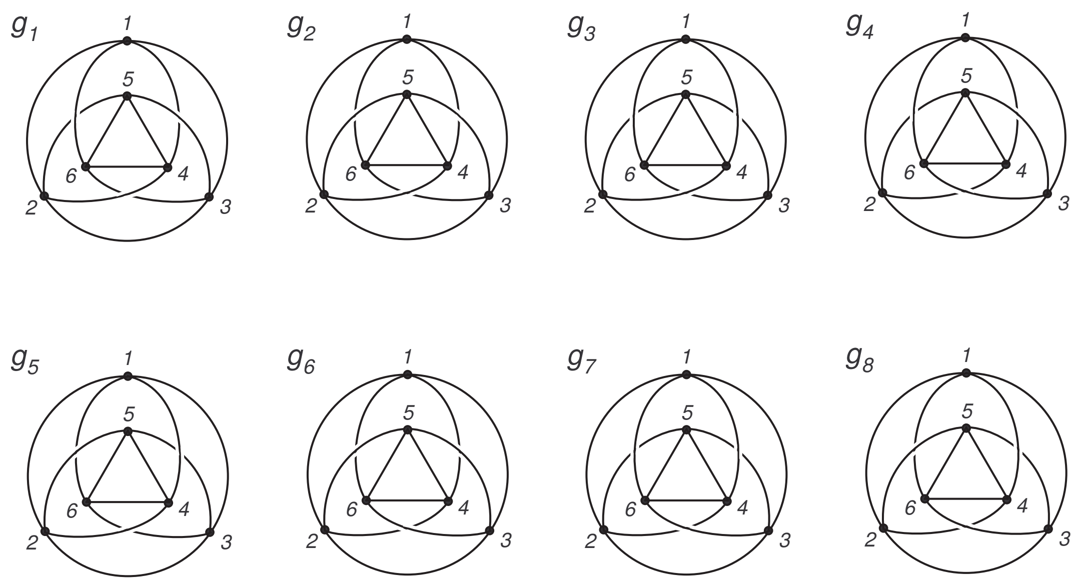

Fig. 1.2.

2. Completely distinguishable projections of planar graphs

A regular projection $\hat{f}$ of a planar graph G is said to be *knotted* [13] if any spatial embedding of G obtained from $\hat{f}$ is non-trivial. Actually, the regular projection $\hat{f}$ of the octahedron graph as in Example 1.1 is an example of the knotted projection. A planar graph is said to be *trivializable* [13] if it has no knotted projections. For example, if G is homeomorphic to the disjoint union of 1-spheres, then G is trivializable. It is known that there exist infinitely many trivializable planar graphs [13], [10], [11]. Then we have the following.

Proposition 2.1. (1) *Let G be a trivializable planar graph. Then a regular projection $\hat{f}$ of G is completely distinguishable if and only if it has no double points.*
(2) *Let G be a non-trivializable planar graph. If a regular projection $\hat{f}$ of G is a non-trivial completely distinguishable projection then it is knotted.*

Actually it is easy to see that any spatial embedding of a graph which can be obtained from a non-trivial completely distinguishable projection of the graph is not ambient isotopic to its mirror image. By this fact, we can prove Proposition 2.1. We note that the converse of Proposition 2.1 (2) is not true, see Examples 2.2 and 2.3.

Example 2.2. Let G be a non-trivializable planar graph and $\hat{f}$ a completely distinguishable projection of G. Then by producing the local parts in $\hat{f}(G)$ as illustrated in Fig. 2.1, we can construct another knotted projection which is not completely distinguishable.

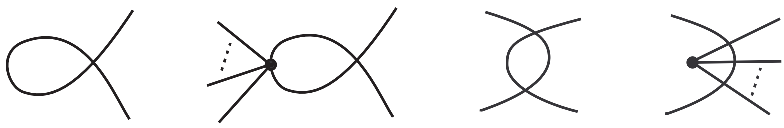

Fig. 2.1.

Example 2.3. Let G be a planar graph and $\hat{f}$ a regular projection of G as illustrated in Fig. 2.2. Since $\hat{f}(G)$ contains a image of the knotted projection of Example 1.1, we have that $\hat{f}$ is knotted. We can also check that $\hat{f}(G)$ does not contain any local parts as in Fig. 2.1. But Fig. 2.3 shows that $\hat{f}$ is not completely distinguishable.

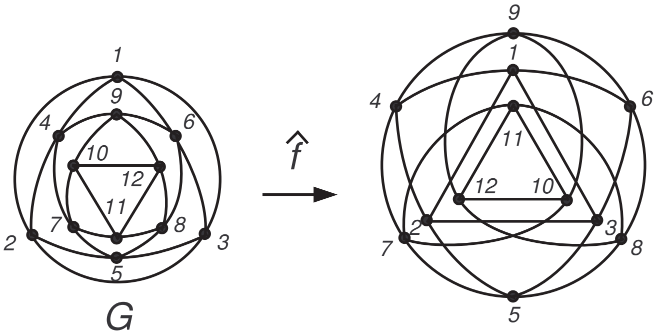

Fig. 2.2.

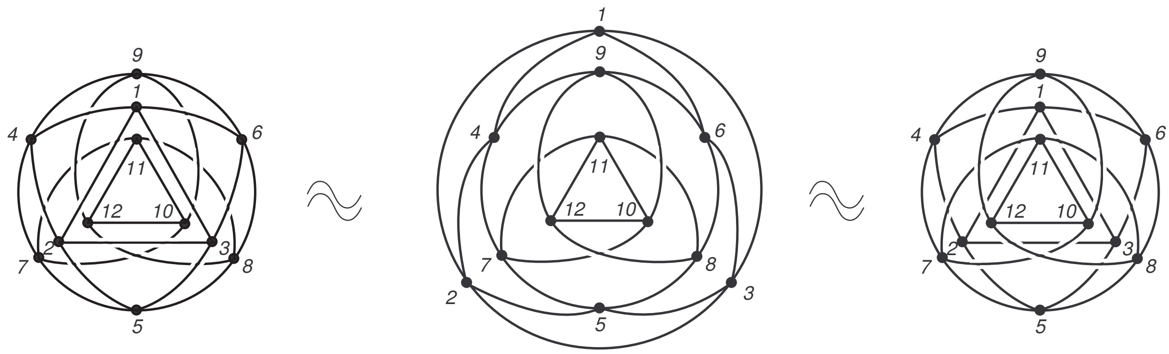

Fig. 2.3.

In this case, let us consider the regular projection of the spatial embedding of G as illustrated in the middle of Fig. 2.3 in the natural way. Then we can see that this regular projection is completely distinguishable.

3. Completely distinguishable projections of non-planar graphs

Let K_n be the *complete graph* on n vertices and $K_{m,n}$ the *complete bipartite graph* on $m+n$ vertices. By Kuratowski's well known theorem [6], we have that K_n is non-planar if $n \geq 5$, and $K_{m,n}$ is non-planar if $\min\{m,n\} \geq 3$.

Theorem 3.1. *Each of K_n and $K_{m,n}$ has a completely distinguishable projection.*

We note that each of K_n for $n \leq 4$ and $K_{m,n}$ for $\min\{m,n\} \leq 2$ is planar and trivializable [10]. So in this case only embeddings from the graph into $\mathbf{R}^2$ are completely distinguishable projections by Proposition 2.1 (1).

To prove Theorem 3.1, we recall the *Wu invariant* [16] of a spatial graph. Let X be a topological space and

$$C_2(X) = \{(x,y) \in X \times X \mid x \neq y\}$$

the *configuration space* of ordered two points of X. Let σ be an involution on $C_2(X)$ defined by $\sigma(x,y) = (y,x)$. Then we call the integral cohomology group of $\mathrm{Ker}(1+\sigma_\#)$ the *skew-symmetric integral cohomology group* of the pair $(C_2(X),\sigma)$ and denote it by $H^*(C_2(X),\sigma)$. It is known that $H^2(C_2(\mathbf{R}^3),\sigma) \cong \mathbf{Z}$ [16]. We denote a generator of $H^2(C_2(\mathbf{R}^3),\sigma)$ by Σ. Let f be a spatial embedding of a graph G. Then f induces a homomorphism

$$(f^2)^* : H^2(C_2(\mathbf{R}^3),\sigma) \longrightarrow H^2(C_2(G),\sigma).$$

We call $(f^2)^*(\Sigma)$ the *Wu invariant* of f and denote it by $\mathcal{L}(f)$. In particular, if G is homeomorphic to K_5 or $K_{3,3}$ then $H^2(C_2(G),\sigma) \cong \mathbf{Z}$ [14] and $\mathcal{L}(f)$ coincides with the *Simon invariant* [12]. Fig. 3.1 illustrates some spatial embeddings of K_5 and $K_{3,3}$ with their Simon invariants under the suitable orientations of edges. We note that $\mathcal{L}(f)$ coincides with twice the *linking number* if G is homeomorphic to the disjoint union of two 1-spheres. We refer the reader to [14] for a diagramatic calculation of $\mathcal{L}(f)$. The generator Σ of $H^2(C_2(\mathbf{R}^3),\sigma)$ depends on the orientation of $\mathbf{R}^3$ [16],

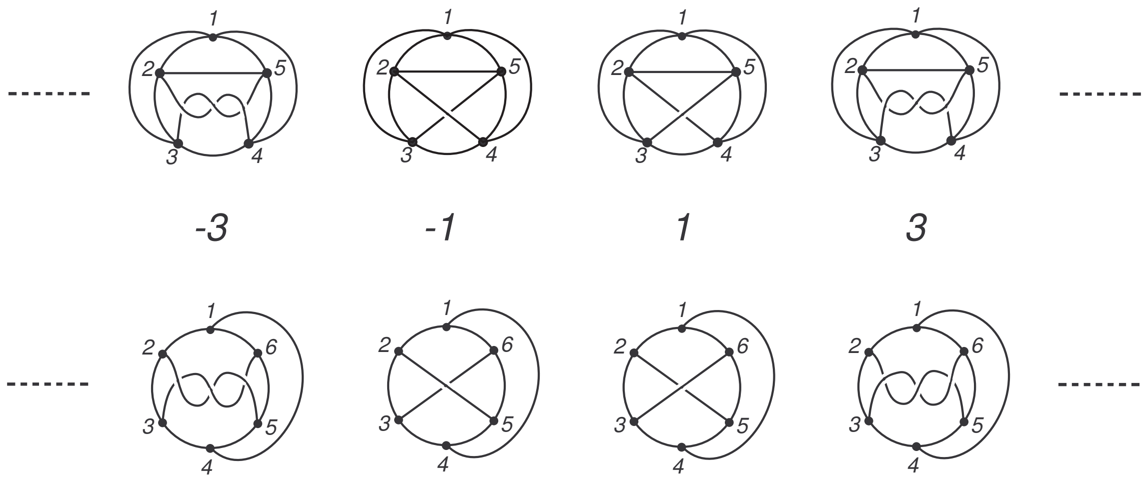

Fig. 3.1.

namely it holds that $\mathcal{L}(f!) = -\mathcal{L}(f)$ for a spatial embedding f of G, where $f!$ denotes the mirror image of f. Since $H^2(C_2(G), \sigma)$ is torsion free [14], we have that if f is ambient isotopic to $f!$ then $\mathcal{L}(f) = 0$. It is known that $\mathcal{L}(f) \neq 0$ for any spatial embedding f of a non-planar graph, namely any spatial embedding of a non-planar graph is not ambient isotopic to its mirror image [9].

In the following we show that K_n has a completely distinguishable projection. Since K_n has a trivial completely distinguishable projection for $n \leq 4$, we assume that $n \geq 5$. We set $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. We denote the edge of K_n connecting v_i and v_j by $e_{i,j}$ $(1 \leq i, j \leq n)$. Let $W_n = C_{n-1} \cup \mathrm{st}(v_n)$ be a subgraph of K_n embedded in $\mathbf{R}^2$ as illustrated in Fig. 3.2, where $C_{n-1} = e_{1,2} \cup e_{2,3} \cup \cdots \cup e_{1,n-1}$ and $\mathrm{st}(v_n) = \cup_{i=1}^{n-1} e_{i,n}$. Then C_{n-1} divides $\mathbf{R}^2$ into two domains and we can construct a regular projection $\hat{f} : K_n \to \mathbf{R}^2$ by embedding each of the edges in $E(K_n) - E(W_n)$ into the non-compact domain such that any pair of the edges $e_{i,j}$ and $e_{k,l}$ $(1 \leq i < k < j < l \leq n-1)$ has exactly one double point and the other pair of the edges has no double points. Let P be the double point between $e_{i,j}$ and $e_{k,l}$ $(1 \leq i < k < j < l \leq n-1)$. Then we can find a subgraph $H = C_{n-1} \cup e_{i,n} \cup e_{k,n} \cup e_{j,n} \cup e_{l,n} \cup e_{i,j} \cup e_{k,l}$ of K_n which is homeomorphic to K_5 such that $\hat{f}|_H$ is a regular projection of H with exactly one double point P. Let g and $g!$ be exactly two spatial embeddings of H which can be obtained from $\hat{f}|_H$. Then we have that $|\mathcal{L}(g)| = |\mathcal{L}(g!)| = 1$. Since $\mathcal{L}(g!) = -\mathcal{L}(g)$, we have that $\mathcal{L}(g) \neq \mathcal{L}(g!)$. This implies that $\hat{f}$ is completely distinguishable. We can also prove that $K_{m,n}$ has a completely distinguishable projection in a similar way by using the Simon invariant of a spatial embedding of $K_{3,3}$ instead of the one of K_5. Fig. 3.3 illustrates a construction of a completely distinguishable projection of $K_{m,n}$.

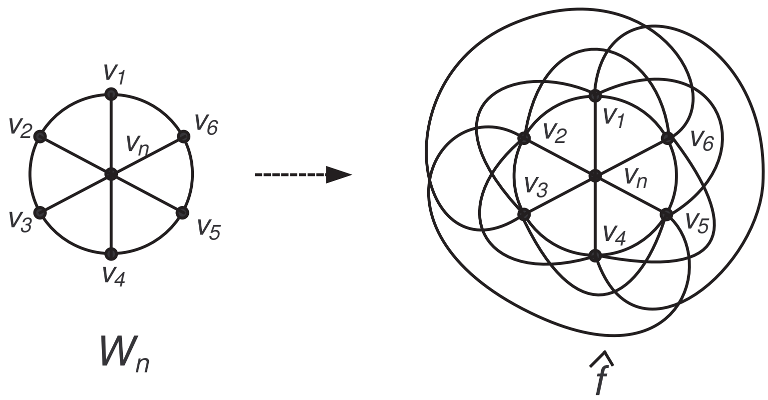

Fig. 3.2. $\hat{f} : K_n \to \mathbf{R}^2$ $(n = 7)$

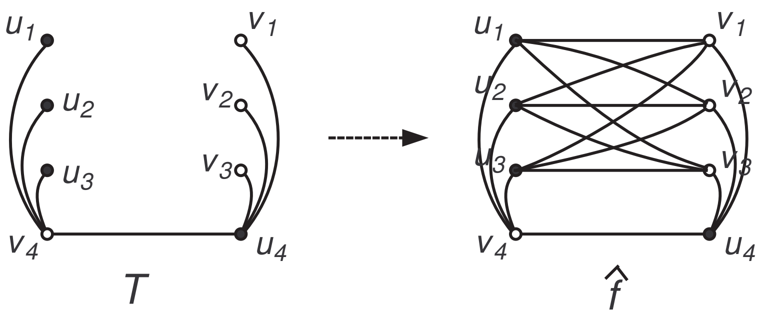

Fig. 3.3. $\hat{f} : K_{m,n} \to \mathbf{R}^2$ $(m = n = 4)$

We remark here that the construction above is based on a complete calculation of the Wu invariants of spatial embeddings of K_n and $K_{m,n}$ by the author, see [8] for the details.

Conjecture 3.2. *Every graph has a completely distinguishable projection.*

4. Minimal crossing projection

Let $c(\hat{f})$ be the number of double points of a regular projection $\hat{f} : G \to \mathbf{R}^2$. Then we call $c(G) = \min\{c(\hat{f}) \mid \hat{f} : G \to \mathbf{R}^2 \text{ is a regular projection}\}$ a *minimal crossing number* of G. A regular projection $\hat{f} : G \to \mathbf{R}^2$ of G is called a *minimal crossing projection* if $c(\hat{f}) = c(G)$. Let $\hat{f}$ be a completely distinguishable projection of K_6 (resp. $K_{3,4}$) constructed by the method in the proof of Theorem 3.1. Since $c(K_6) = 3$ and $c(K_{3,4}) = 2$ (cf. [4]), we can see that $\hat{f}$ is not a minimal crossing projection. Then we ask the following.

Question 4.1. (M. Ozawa) *Is a minimal crossing projection a completely distinguishable projection?*

Note that a minimal crossing projection of a planar graph is an embedding of the graph into $\mathbf{R}^2$, namely a trivial completely distinguishable projection. Besides we can give an affirmative answer to Question 4.1 for the case of minimal crossing number one [9].

We present a piece of circumstantial evidence as follows. Let $\hat{f}$ be the regular projection of K_{2n}, which is Blažek-Koman's construction of a candidate for the minimal crossing projection of K_{2n} [1]. Fig. 4.1 illustrates the case of $n = 4$ and in this case $\hat{f}$ is a minimal crossing projection of K_8.[1] Then we can see that $\hat{f}$ is completely distinguishable in a similar way as the proof of Theorem 3.1.

Next, let $\hat{f}$ be the regular projection of $K_{m,n}$, which is Zarankiewicz's construction of a candidate for the minimal crossing projection of $K_{m,n}$ [17]. Fig. 4.2 illustrates the case of $m = n = 6$ and in this case $\hat{f}$ is

[1] R. K. Guy conjectured in [2] that

$$c(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

and showed that the conjecture above is true for $n \leq 10$ [3].

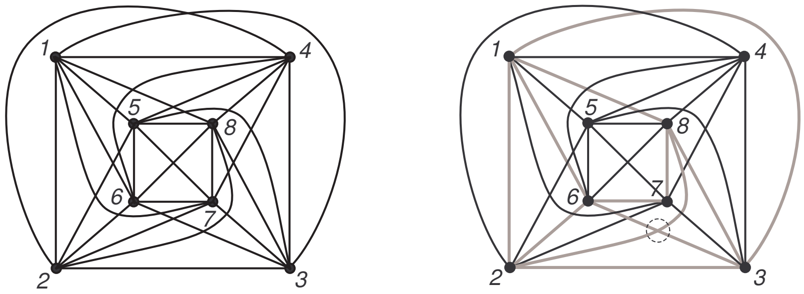

Fig. 4.1. $\hat{f} : K_{2n} \to \mathbf{R}^2$ $(n = 4)$

a minimal crossing projection of $K_{6,6}$.[2] Then we can also see that $\hat{f}$ is completely distinguishable in a similar way as the proof of Theorem 3.1.

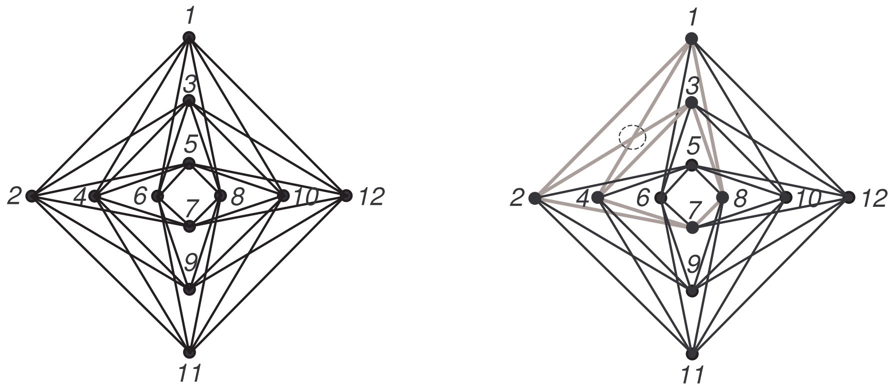

Fig. 4.2. $\hat{f} : K_{m,n} \to \mathbf{R}^2$ $(m = n = 6)$

Conjecture 4.2. *Every minimal crossing projection is completely distinguishable.*

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[2] From the beginning, the crossing number problem for $K_{m,n}$ is called *Turán's brick factory problem* (cf. [15]). In [17], K. Zarankiewicz claimed that

$$c(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

However Guy pointed out a gap for Zarankiewicz's proof [2], and now this is called Zarankiewicz's conjecture. At present, Zarankiewicz's conjecture is true for $m \leq 6$ [5].

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