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曲面に接した空間グラフのトポロジー
Topology of spatial graphs attaching to a surface

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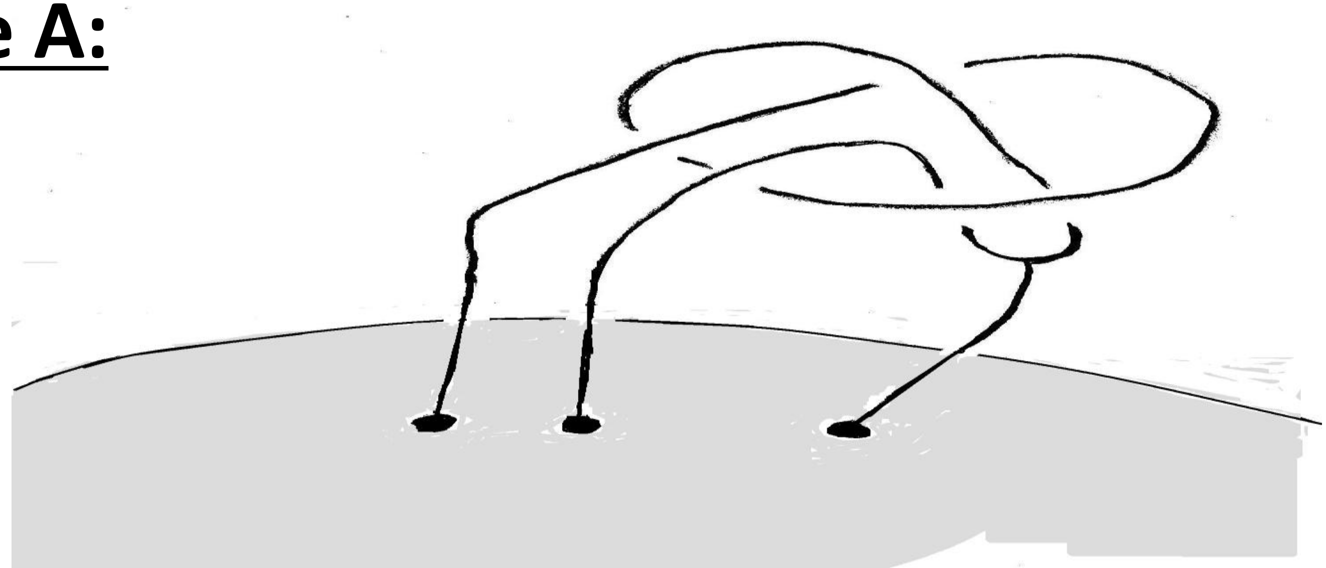
References for this special topics

[1] A. Kawauchi, On a complexity of a spatial graph. in: Knots and soft-matter physics, Topology of polymers and related topics in physics, mathematics and biology, Bussei Kenkyu 92-1 (2009-4), 16-19.

[2] A. Kawauchi, On transforming a spatial graph into a plane graph,in: Statistical Physics and Topology of Polymers with Ramifications to Structure and Function of DNA and Proteins, Progress of Theoretical Physics Supplement, No. 191(2011), 235-244.

[3] A. Kawauchi, Spatial graphs attaching to a surface, in preparation.

Example A:



***Question*. In what sense , this object is "knotted" or "unknotted" ?**

In this talk, the answer will be "β-unknotted" but "knotted", "γ-knotted" and "Γ-knotted" under some definitions introduced from now.

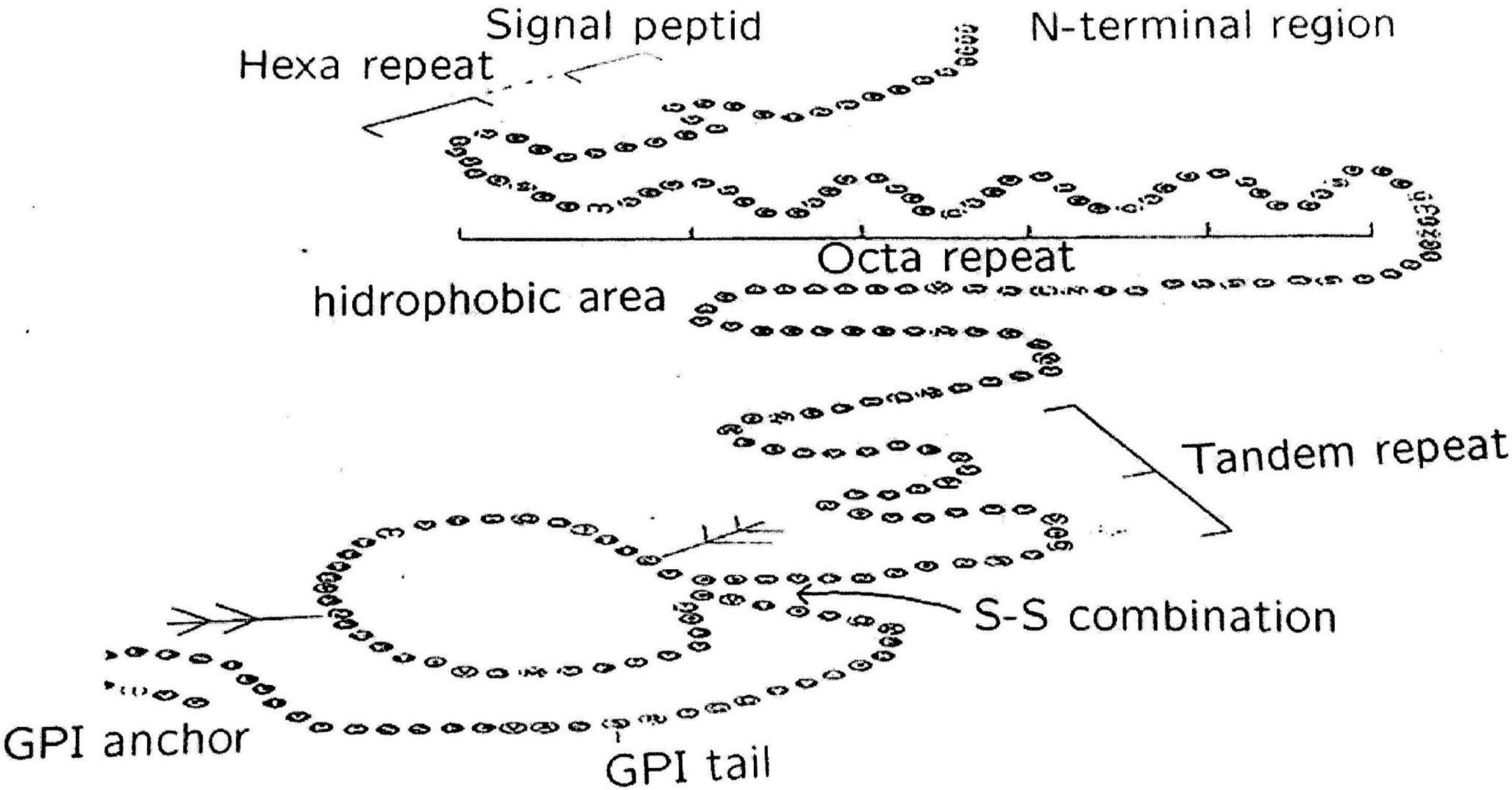
<u>Example B:</u> Proteins attached to a cell surface

Some points of S. B. Prusiner's theory are:

(1) By losing the N-terminal region, Prion precursor protein changes into Cellular PrP (PrP^c) or Scrapie PrP (PrP^{SC}), and α-helices change into β-sheets.
(2) The conformations of PrP^c and PrP^{SC} may differ although the linear structures are the same.
(3) There is one S-S combination.

- Z. Huang et al., Proposed three-dimensional Structure for the cellular prion protein, Proc. Natl. Acad. Sci. USA, 91(1994), 7139-7143.
- K. Basler et al., Scrapie and cellular PrP isoforms are encoded by the same chromosomal gene, Cell 46(1986), 417-428.

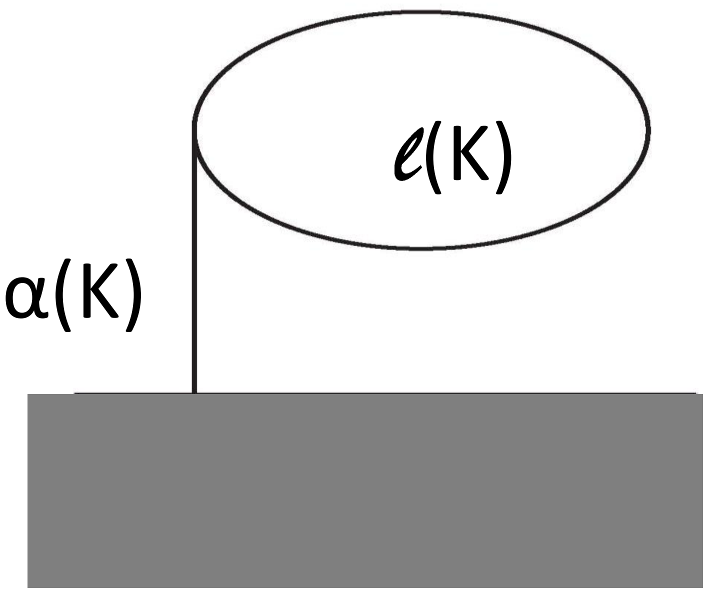
Prion Precursor Protein

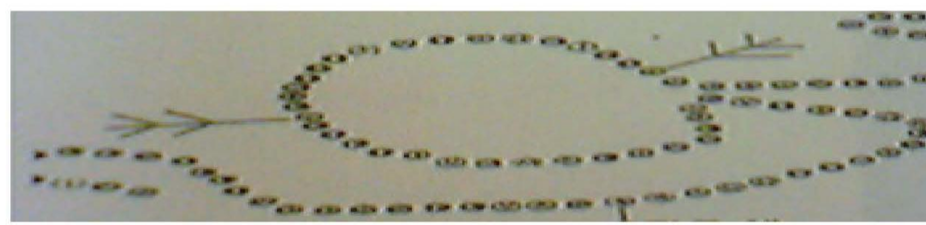


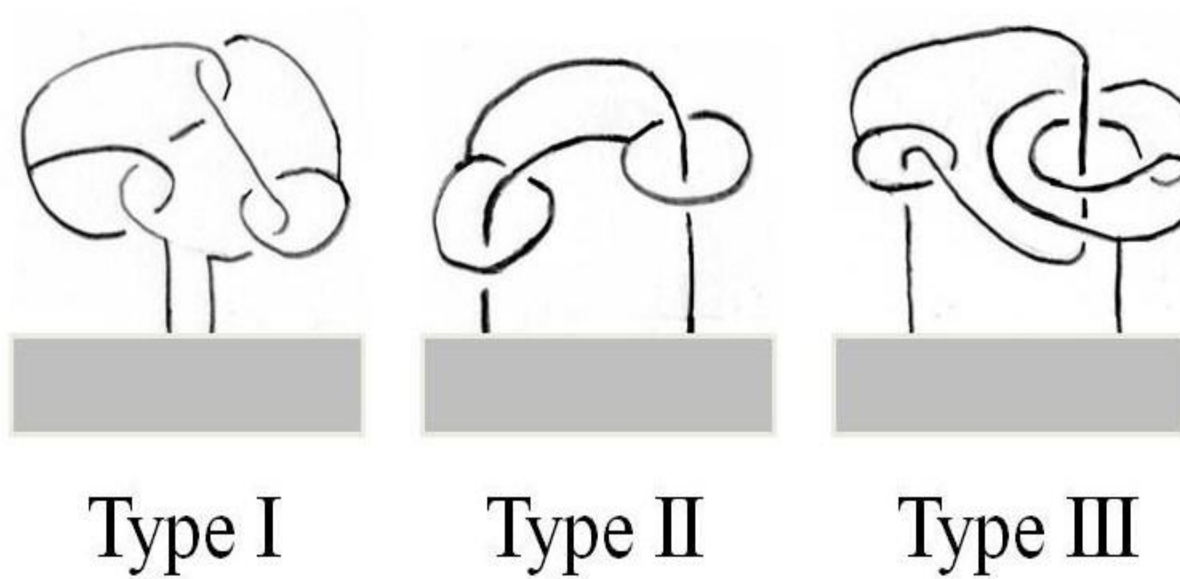
From:

K. Yamanouchi & J. Tateishi Editors, Slow Virus Infection and Prion (in Japanese), Kindaishuppan Co. Ltd. (1995)

<u>Definition.</u> **A <u>prion-string</u> is a spatial graph $K = \ell(K) \cup \alpha(K)$ in the upper half space $\mathbb{H}^3$ consisting of <u>S-S loop</u> $\ell(K)$ and <u>GPI-tail</u> $\alpha(K)$ joining the S-S vertex in $\ell(K)$ with the <u>GPI-anchor</u> in $\partial\mathbb{H}^3$.**





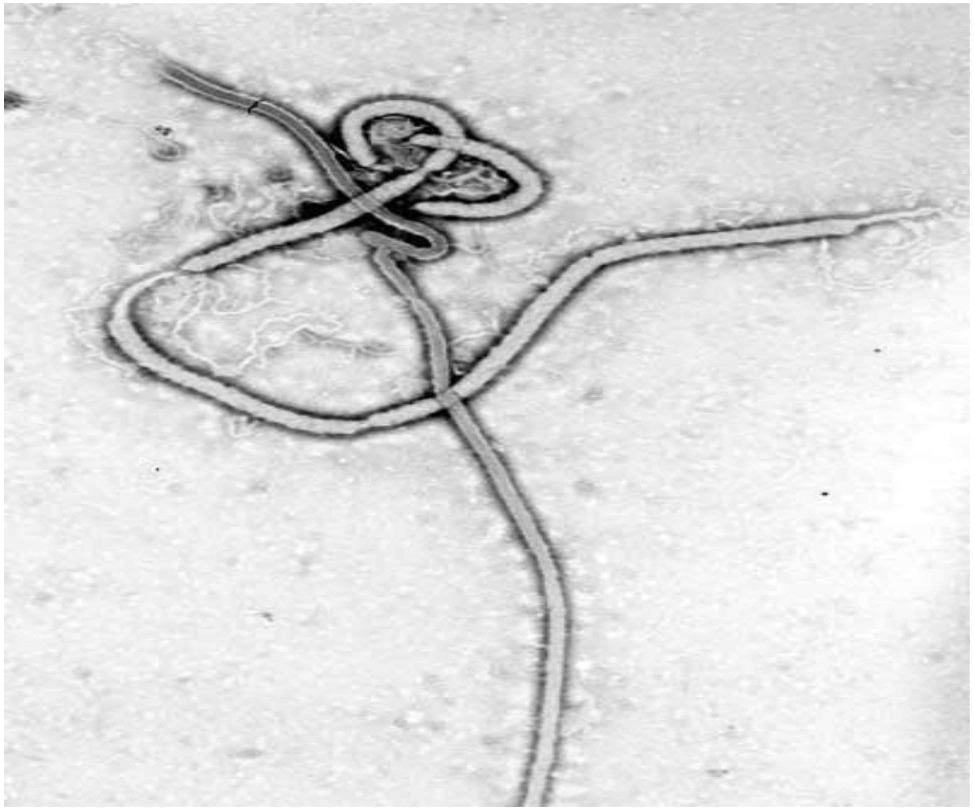


Topological models of prion-proteins
（cf.　[J. Math. System Sci. 2012]）

[J. Math. System Sci. 2012]
A. Kawauchi and K. Yoshida, Topology of prion proteins, Journal of Mathematics and System Science 2(2012), 237-248.

<u>Example C</u>: A string-shaped virus

A virus of EBOLA haemorrhagic fever



http://www.scumdoctor.com/Japanese/disease-prevention/infectious-diseases/virus/ebola/Pictures-Of-The-Effects-Of-Ebola.html

1. Several notions on unknotted graphs

1.1. A based diagram and a monotone diagram

Let Γ be a graph without degree one vertices, and $G = G(\Gamma)$ a spatial graph in R^3. Let Γ_i $(i=1,2,\ldots,r)$ be an ordered set of the components of Γ, and $G_i = G(\Gamma_i)$ the corresponding spatial subgraph of $G = G(\Gamma)$. Let T_i be a **maximal tree** of G_i.

Note: We consider a topological graph without degree 2 verticies, so that $T_i = \phi$ if G_i is a knot or link, and T_i = one vertex if G_i has just one vertex (of degree $\geqq 3$).

Let $T = T_1 \cup T_2 \cup \ldots \cup T_r$. Call it a **base** of G.

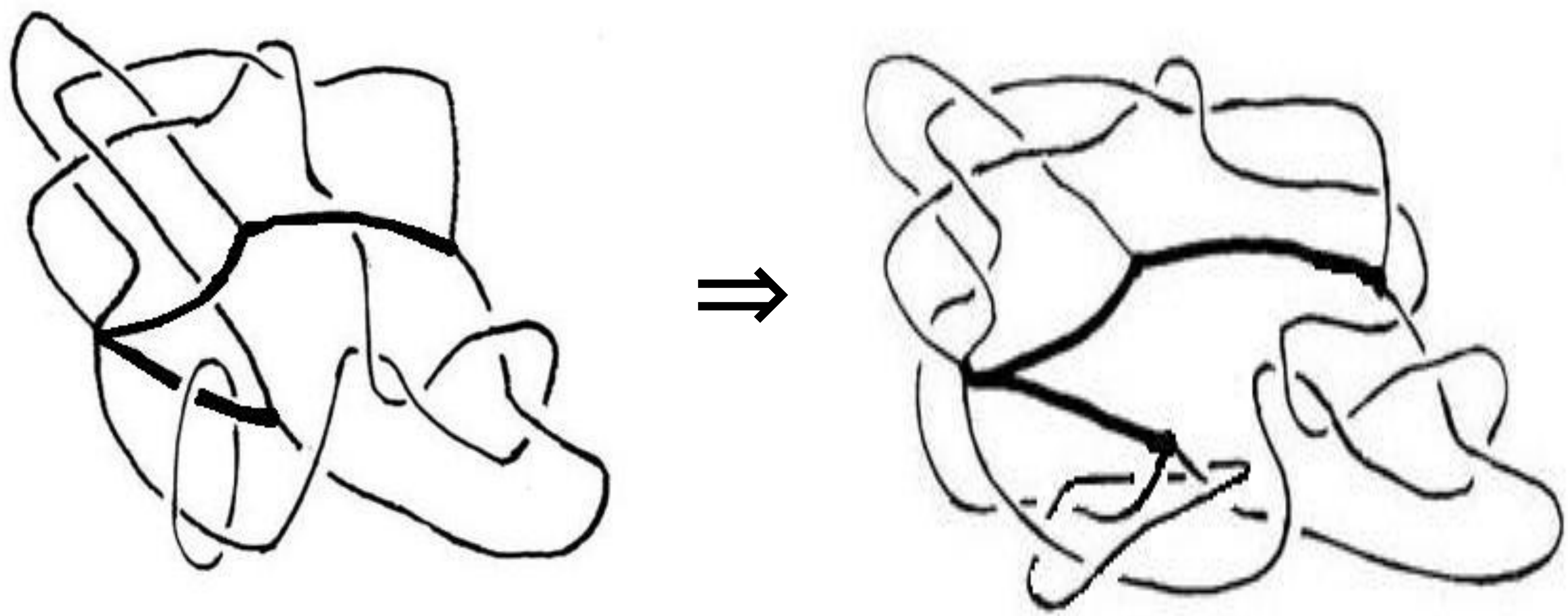
Note: There are only finitely many bases of G.

G is obtained from a basis T by attaching **edges** (i.e., arcs or loops) to T.

Let D be a diagram of a spatial graph $G=G(\Gamma)$, and D_T the sub-diagram of D corresponding to T.

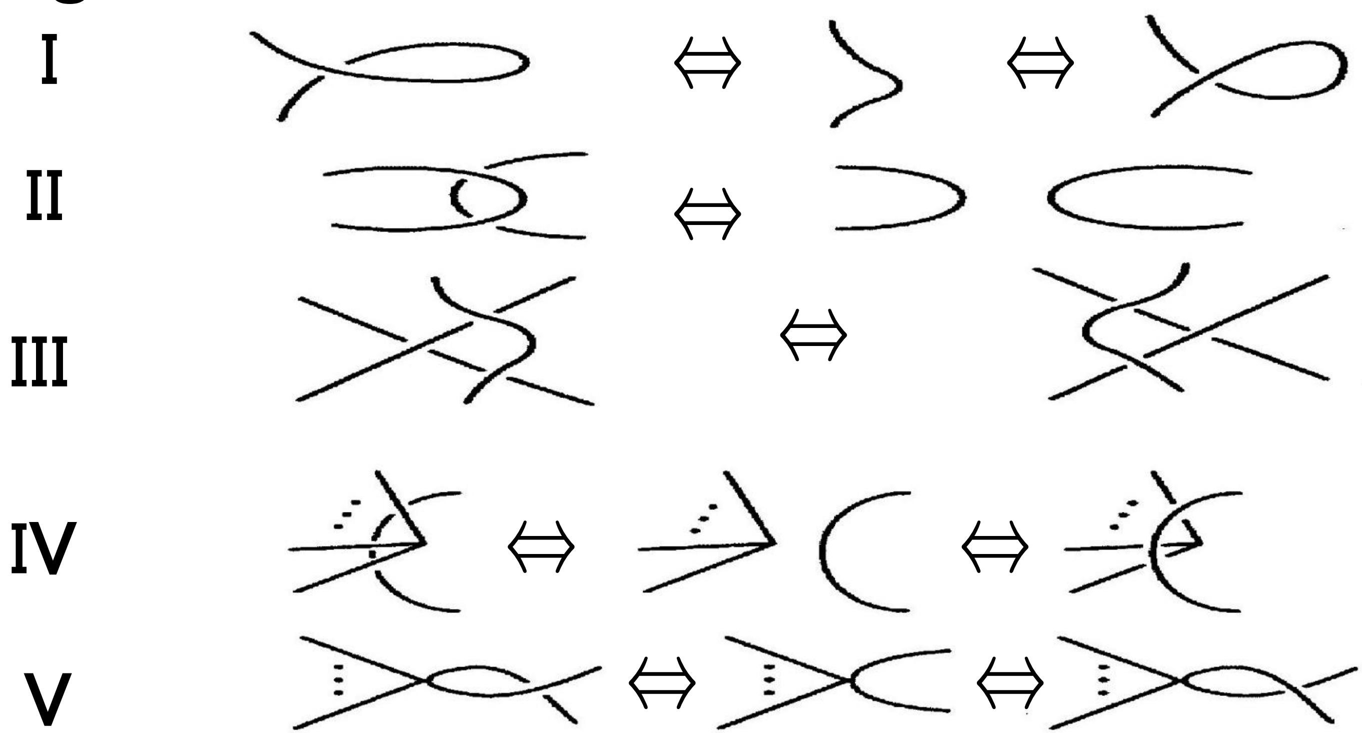
Let $c_D(D_T)$ be the number of crossing points of D whose upper or lower crossing points belong to D_T.

Definition. D is *a* ***based diagram*** (on base T), written as (D;T) if $c_D(D_T)=0$.



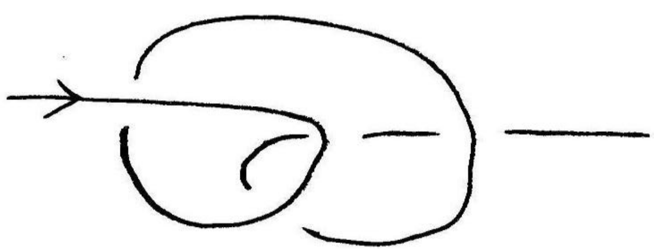
Lemma. For $\forall$ base T of G, $\forall$ diagram D of G is deformed into a based diagram on T by generalized Reidemeister moves.

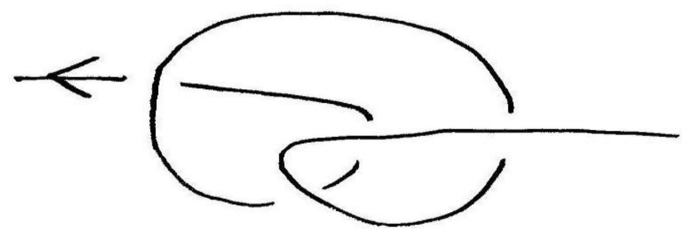
The generalirez Reidemeister moves:



Let α be an edge of G=G(Γ) attaching to a base T.

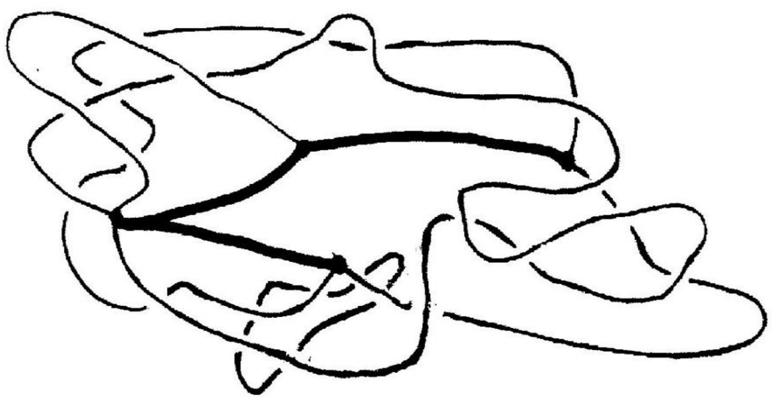
<u>Definition</u>. An edge diagram D_α in a diagram D of G is **<u>monotone</u>** if:

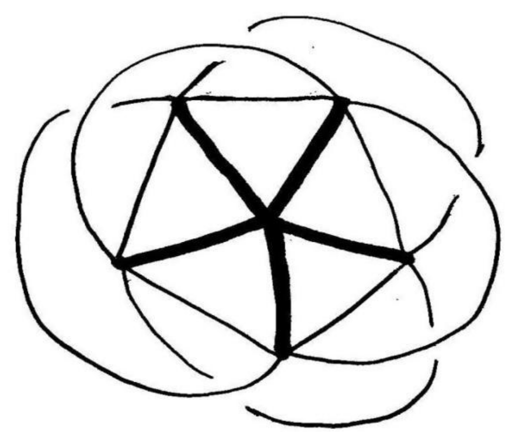




A sequence on the edges of a based graph (G ,T) is **regularly ordered** if an order on the edges such that any edge belonging to G_i is smaller than any edge belonging to G_j for $i<j$ is specified.

Definition. A based diagram (D;T) is **monotone** if there is a regularly ordered edge sequence α_i (i=1,2,⋯,m) of (G ,T) such that D_{α_i} is monotone and D_{α_i} is upper than D_{α_j} for i<j .





1.2. Complexity

Definition.

The **warping degree** d(D;T) of a based diagram (D;T) is the least number of crossing changes on edge diagrams attaching to T needed to obtain a monotone diagram from (D;T).

The **crossing number** of (D;T) is denoted by c(D;T).

If D is a knot or link diagram or an edge diagram, then the warping degree and crossing number of D are denoted by d(D) and c(D), respectively.

A similar notion for a knot or link is given in :

[Lickorish-Millett 1987] W. B. R. Lickorish and K. C. Millett, A polynomial invariant of oriented links, Topology 26(1987), 107-141.

[Fujimura 1988] S. Fujimura, On the ascending number of knots, thesis, Hiroshima University, 1988.

[Fung 1996] T. S. Fung, Immersions in knot theory, a dissertation, Columbia University, 1996.

[Kawauchi 2007] A. Kawauchi, Lectures on knot theory (in Japanese), Kyoritu Shuppan, 2007.

[Ozawa 2010] M. Ozawa, Ascending number of knots and links. J. Knot Theory Ramifications 19 (2010), 15-25.

[Shimizu 2010] A. Shimizu, The warping degree of a knot diagram, J. Knot Theory Ramifications 19(2010), 849-857.

Properties of the warping degree

(1) For the warping degree $\vec{d}$ of an *oriented* edge diagram D_α,

$$\vec{d}(D_\alpha) + \vec{d}(-D_\alpha) = c(D_\alpha),$$

$$d(D_\alpha) = \min\{\vec{d}(D_\alpha), \vec{d}(-D_\alpha)\}.$$

Example. d() =1, for

$\vec{d}$ ()=1, $\vec{d}$ ()=3.

(2) [**Shimizu** 2010]

For an oriented knot diagram D,

$$\vec{d}(D)+\vec{d}(-D) \leqq c(D)-1,$$

where the equality holds if and only if D is an alternating diagram.

Definition.

The **complexity** of a based diagram (D;T) is the pair cd(D;T)= (c(D;T), d(D;T)) together with the dictionary order.

d(D;T)$\leqq$c(D;T) implies:

Note (A. Shimizu).

The dictionary order on cd(D;T) is equivalent to the numerical order on $c(D;T)^2+d(D;T)$.

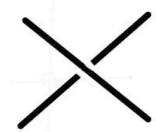
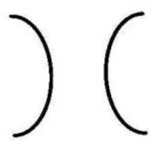
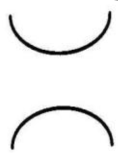
Definition.

The **complexity** of a spatial graph G is

$$\gamma(G) = \min\{cd(D;T) \mid (D;T) \in [D_G]\}$$

(in the dictionary order).

Let $\gamma(G) = (c_\gamma(G), d_\gamma(G))$.

Our basic viewpoint of complexity. This complexity is reducible by a crossing change  ⇔  or a splice  ⇒  or  until we obtain a graph in a plane.

(1) If $d_\gamma(G)>0$, then $\exists$ G’ with $\gamma(G') < \gamma(G)$ by a crossing change.

$d_\gamma(G)=0 \Leftrightarrow$ G is equivalent to G’ with a monotone diagram (D’;T’) with $c_\gamma(D';T')=c_\gamma(G)$.

(2) If $c_\gamma(G)>0$, then $\exists$ G’ with $\gamma(G') < \gamma(G)$ by a splice.

$c_\gamma(G)=0 \Leftrightarrow$ G is equivalent to a graph in a plane.

1.3. The warping degree and an unknotted graph

Definition.

The **warping degree** of G is :

$$d(G) = \min\{d(D;T) \mid (D;T) \in [D_G]\}$$

Definition.

G is **unknotted** if d(G)= 0.

When Γ consists of loops,

G is unknotted ⇔ G is a trivial link.

Assume Γ has a vertex of degree $\geqq 3$.

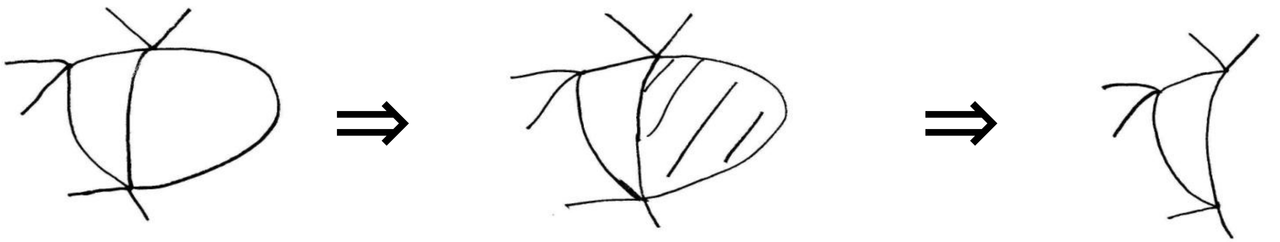
<u>Lemma 1.3.1</u>. For $\forall$ G, $\exists$ finitely many crossing changes on G to make G with d(G)=0.

<u>Lemma 1.3.2</u>. For $\forall$ given graph Γ, $\exists$ only finitely many G of Γ with d(G)=0 up to equivalences.

<u>Lemma 1.3.3</u>. If d(G)=0, then $\exists$ T such that

G/T is equivalent to $S^1 \vee S^1 \vee \dots \vee S^1 \subset R^2$.

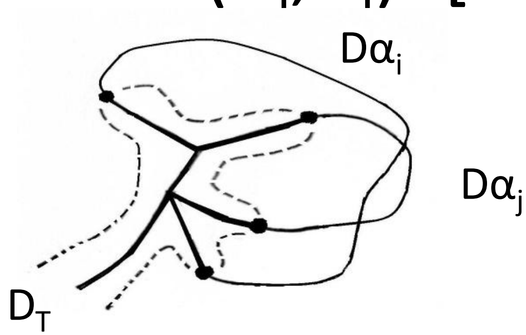
Lemma 1.3.4. A connected G with d(G)=0 is deformed into a basis T by a sequence of edge reductions:



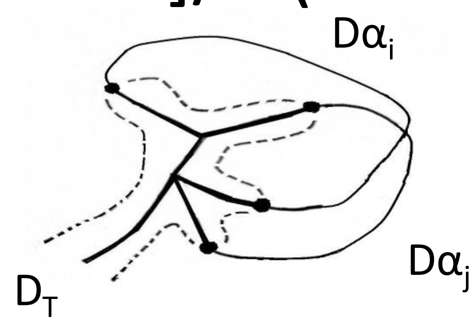
Corollary 1.3.5. For ∀ G with d(G)=0, ∃ T such that every edge (arc or loop) attaching to T is in a trivial constituent knot.

Given D_T , the **cross index** of α_i and α_j $(i\neq j)$:

$$\varepsilon(\alpha_i,\alpha_i)=[1-(-1)^{\#(D\alpha i \cap D\alpha j)}]/2 \ (=0 \text{ or } 1).$$



cross index =0



cross index =1

The **total cross index** of Γ on D_T :

$$\varepsilon(\Gamma;\, D_T) = \textstyle\sum_{i<j} \varepsilon(\alpha_i,\alpha_j).$$

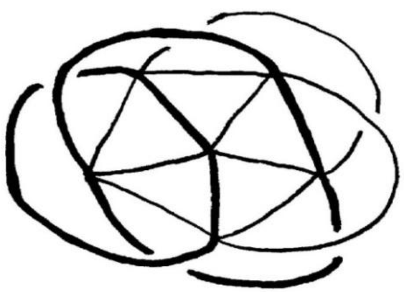
Lemma 1.3.6. Let $d(G)=0$. Then

$$\min\{c(D;T) \mid (D;T)\in [D_G],\ d(D;T)=0\} = \varepsilon(\Gamma;\, D_T).$$

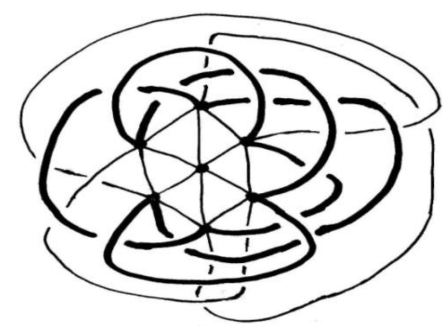
Conway-Gordon Theorem.

Every spatial 6-complete graph K_6 contains a non-trivial constituent link.

Every spatial 7-complete graph K_7 contains a non-trivial constituent knot.



An unknotted K_6



An unknotted K_7

1.4. The γ-warping degree and a γ-unknotted graph

Definition.

The γ-**warping degree** of G is the number $d_\gamma(G)$ for the complexity $\gamma(G) = (c_\gamma(G), d_\gamma(G))$ of G.

Definition. G is γ-**unknotted** if $d_\gamma(G) = 0$.

γ-unknotted ⇒ unknotted

1.5. A Γ-unknotted graph and the (γ,Γ)-warping degree

Let $\gamma(\Gamma) = \min\{\gamma(G) \mid G \text{ is a spatial graph of } \Gamma\}$.

Definition.

A Γ-**unknotted** graph G is a spatial graph of Γ with $\gamma(G) = \gamma(\Gamma)$.

Note.

(1) Let $\gamma(\Gamma) = (c_\gamma(\Gamma), d_\gamma(\Gamma))$. Then $d_\gamma(\Gamma)=0$.

Γ-unknotted $\Rightarrow$ γ-unknotted $\Rightarrow$ unknotted.

(2) $c_\gamma(\Gamma)=0$ if and only if Γ is a plane graph.

(3) A spatial plane graph G is Γ-unknotted

$\Leftrightarrow$ G is equivalent to a graph in a plane.

Definition.

$O \quad$ = {unknotted graphs of Γ}.

O_γ^G = {γ-unknotted graphs on $(D;T) \in [D_G]$ with $cd(D;T)=\gamma(G)$}.

O_γ = $\cup \{O_\gamma^G \mid G$ is a spatial graph of $\Gamma\}$
= {γ-unknotted graphs of Γ}.

O_Γ = {Γ-unknotted graphs}.

Then $O \supset O_\gamma \supset O_\Gamma$.

Note: $O_\gamma^G \subset O_\Gamma$ or $O_\gamma^G \cap O_\Gamma = \phi$ for every G.

Definition.

The (γ,Γ)-**warping degree** $d_\gamma^\Gamma(G)$ of G is:

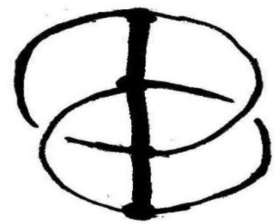
$$d_\gamma^\Gamma(G) = d_\gamma(G) + \rho(O_\gamma^G, O_\Gamma).$$

(ρ denotes the Gordian distance.)

By definition, $d(G) \leqq d_\gamma(G) \leqq d_\gamma^\Gamma(G)$.

$d_\gamma^\Gamma(G) = 0$ if and only if G is Γ-unknotted.

1.6. Examples

Example 1.6. 1. Let G =  .

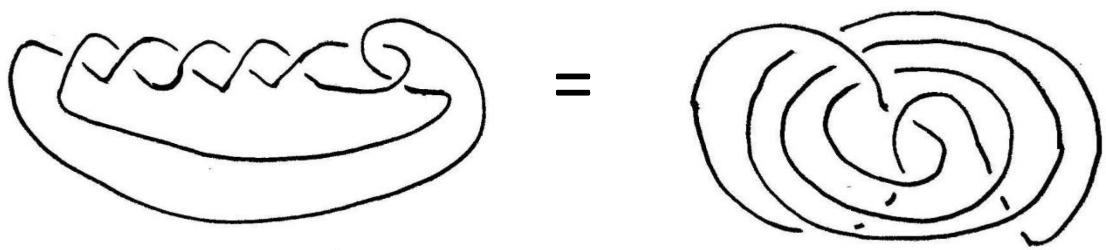
G has $c_\gamma(G)=2$, for G has a Hopf link as a constituent link.

$d(G)=d_\gamma(G)= 0$.

Because G is a planar graph, if G is Γ-unknotted, then $c_\gamma(G)=0$, a contradiction.

Hence $d^{\Gamma}_{\gamma}(G) =1$.

Lemma 1.6.2. (1) (**[Fung 1996]** , **[Ozawa 2010]**)
If K is a knot with d(K)=1, then K is a non-trivial twist knot.



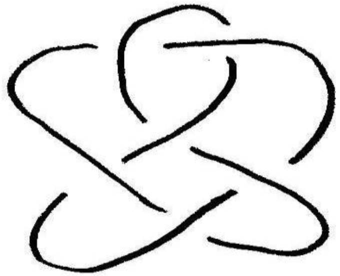
(2) If G is a θ-curve with d(G)=1, then the 3 constituent knots of G consist of two trivial knots and one non-trivial twist knot .

Example 1.6.3. (([Fung 1996] , [Ozawa 2010], [Shimizu 2010])

For K=  5_2 , we have

$c_\gamma(K)=5$, $d(K)=1< d_\gamma(K)= d^\Gamma_\gamma(K)=2$.

Example 1.6.4.

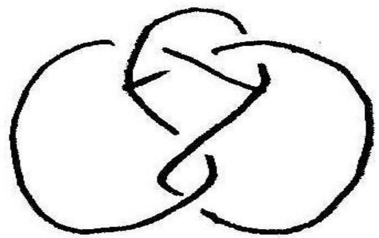
For K =  6_2,

$c_\gamma(K)=6$, $d(K)=d_\gamma(K)=d^\Gamma_\gamma(K)=2$.

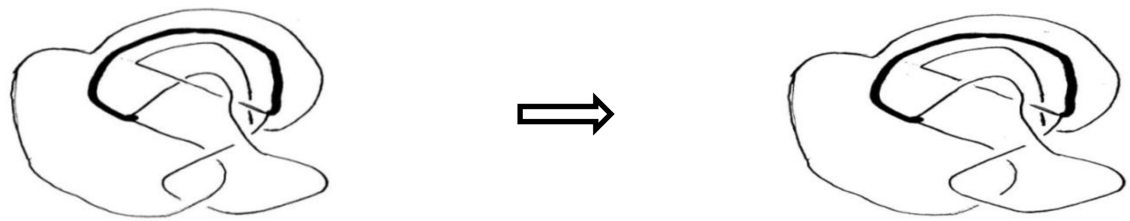
In fact, $d^\Gamma_\gamma(K)\leqq 2$:

By Lemma, $d(K)\geqq 2$ (, for K is not any twist knot).

<u>Example 1.6.5.</u> (Kinoshita's θ-curve)

For G = , we have

$c_\gamma(G) = 7$ and $d(G) = d_\gamma(K) = d^\Gamma_\gamma(G) = 2$.



a based diagram of G a monotone diagram

$O^G_\gamma = O_\Gamma$ implies $\rho(O^G_\gamma, O_\Gamma)=0$. Hence $d_\gamma(G)= d^\Gamma_\gamma(G)$.
Since G is non-trivial and the 3 constituent knots are trivial, we have $d(G) \geqq 2$ by Lemma.
Hence, if $c_\gamma(G)=7$, then $d(G)= d_\gamma(G)= d^\Gamma_\gamma(G)=2$.

By the diagram, $c_\gamma(G) \leqq 7$. We show $c_\gamma(G) \geqq 7$.
By the classification of algebraic tangles with crossing numbers $\leqq 6$ in:

[Moriuchi 2008] H. Moriuchi, Enumeration of algebraic tangles with applications to theta-curves and handcuff graphs, Kyungpook Math. J. 48(2008), 337-357

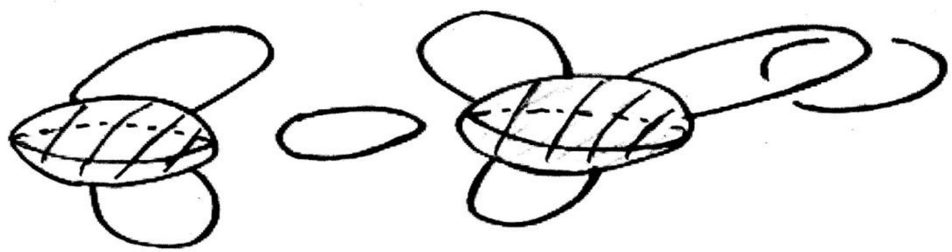
the Kinoshita's θ-curve G cannot have any based diagram with crossing number $\leqq 6$.
Hence $c_\gamma(G)=7$.

1.7. A β-unknotted graph

For a base $T = T_1 \cup T_2 \cup \ldots \cup T_r$ of G, let B be the disjoint union of mutually disjoint 3-ball neighborhoods B_i of T_i in S^3 (i=1,2,...,r).

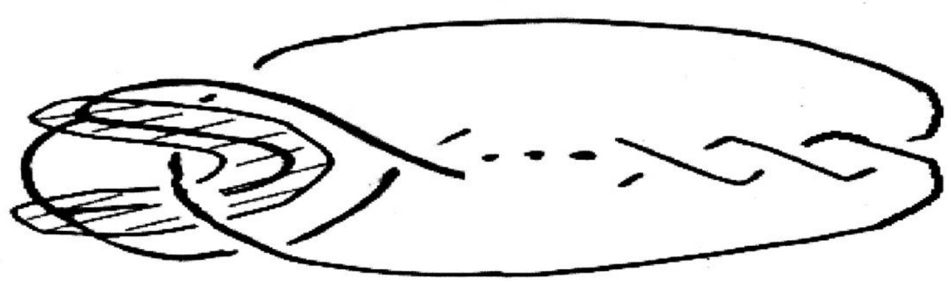
Let $B^c = cl(S^3 - B)$ be the complement domain of B with $L = B^c \cap G = a_1 \cup a_2 \cup \ldots \cup a_n$ an n-string tangle in B^c, called the **complementary tangle** of T.

<u>Definition</u>. G is **<u>β-unknotted</u>** if ∃ a base T of G whose complementary tangle (B^c, L) is trivial.



A trivial complementary tangle

<u>Example 1.7.1.</u> For a θ-curve Γ, ∃ ∞-many β-unknotted graphs G of Γ up to equivalences.



<u>Example 1.7.2.</u> Triviality of the complementary tangle (B^c,L) depends on a choice of a base.

<u>Example 1.7.3.</u> If G is β-unknotted, then G is a **<u>free graph</u>** (i.e., $\pi_1(R^3\text{-}G)$ is a free group), but the converse is not true.

A free β-knotted graph

By definitions and examples explained above, we have:

Theorem.

Γ-unknotted⇒γ-unknotted⇒unknotted

⇒ β-unknotted ⇒ free.

These concepts are mutually distinct.

Note: Given a Γ, ∃ only finitely many Γ-unknotted, γ-unknotted, or unknotted graphs of Γ.

2. Several notions of unknotting numbers of a spatial graph

2.1. The unknotting number

Let O = {unknotted graphs of Γ}.

Definition.

The **unknotting number** u(G) of a spatial graph G of Γ is the distance from G to O by crossing changes on edges attaching to a base:

$$u(G) = \rho(G,O).$$

2.2. A β-unknotting number

Let O_β = {β-unknotted graphs of Γ}.

Definition.

The **β-unknotting number** $u_\beta(G)$ of a spatial graph G of Γ is the distance from G to O_β by crossing changes on edges attaching to a base:

$$u_\beta(G) = \rho(G, O_\beta).$$

2.3. A γ-unknotting number

Given G, let

$$\{D_{G,\gamma}\} = \{(D;T) \in [D_G] \mid c(D;T) = c_\gamma(G)\}$$

(the set of minimal crossing based diagrams).

Definition.

The **γ-unknotting number** $u_\gamma(G)$ of a spatial graph G of Γ is the distance from $\{D_{G,\gamma}\}$ to O by crossing changes on edges attaching to a base:

$$u_\gamma(G) = \rho(\{D_{G,\gamma}\}, O).$$

Note. G is γ-unknotted ⇔ $u_\gamma(G) = 0$.

2.4. Γ-unknotting number

Let O_Γ={Γ-unknotted graphs}.

Definition.

The Γ-**unknotting number** $u^\Gamma(G)$ of G is the distance from G to O_Γ by crossing changes on edges attaching to a base:

$$u^\Gamma(G) = \rho(G, O_\Gamma)$$

Definition.

The (γ,Γ)-**unknotting number** $u_\gamma^\Gamma(G)$ of G is the distance from $\{D_{G,\gamma}\}$ to O_Γ by crossing changes on edges attaching to a base:

$$u_\gamma^\Gamma(G) = \rho(\{D_{G,\gamma}\}, O_\Gamma).$$

2.5. Dsitinctness of the unknotting numbers

Theorem 2.5.1. The unknotting numbers

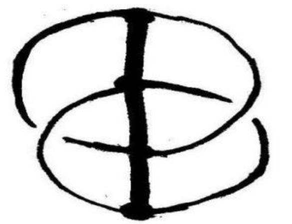
$$u_{\beta}(G),\ u(G),\ u^{\Gamma}(G),\ u_{\gamma}(G),\ u^{\Gamma}_{\gamma}(G)$$

of $\forall$ spatial graph G of $\forall$ graph Γ are mutually distinct topological invariants and satisfy the following inequalities :

$$u_{\beta}(G) \leqq u(G) \leqq \{u_{\gamma}(G), u^{\Gamma}(G)\} \leqq u^{\Gamma}_{\gamma}(G).$$

Proof. The inequalities are direct from definitions. We show that these invariants are distinct.

(1)

G= 

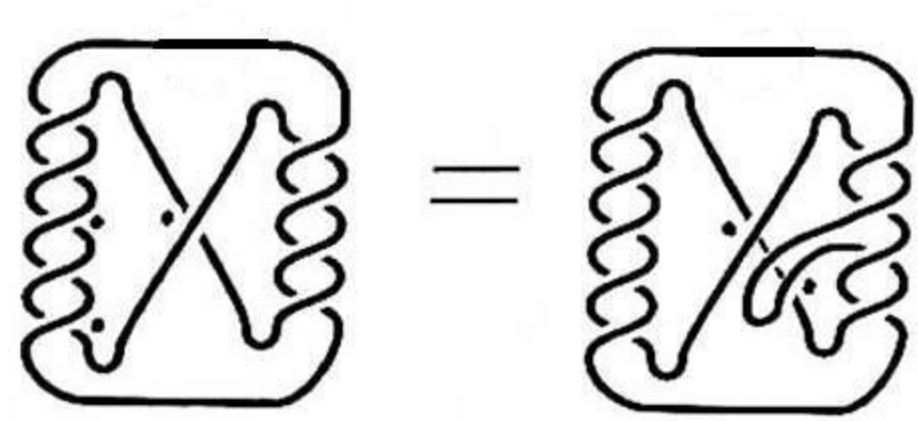
G has $c_\gamma(G)=2$ and hence $u_\beta(G)=u(G)=u_\gamma(G)=0$.

On the other hand, we have

$$u^\Gamma(G)=u^\Gamma_\gamma(G)=1,$$

for G is a spatial graph of a plane graph with a Hopf link as a constituent link and hence not Γ-unknotted.

(2)

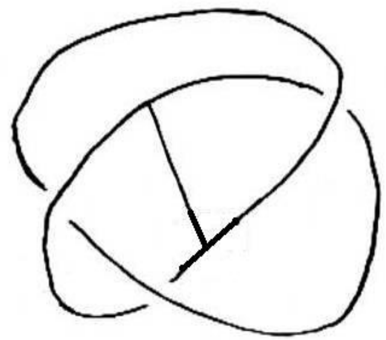
Let G= 

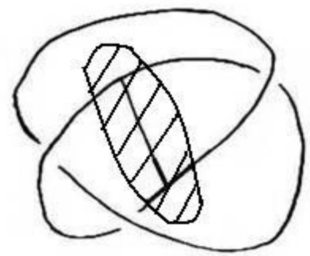
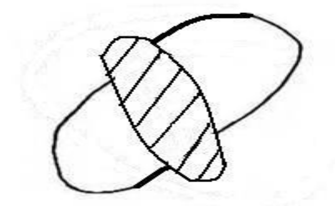
$G=10_8$ has $u(10_8)=2$ and $u_\gamma(10_8)=3$ by **[Nakanishi 1983]** and **[Bleiler 1984]** . Hence

$$u_\beta(G)= u(G) =u^\Gamma(G)=2< u_\gamma(G)=u^\Gamma_\gamma(G)=3.$$

[Nakanishi 1983] Y. Nakanishi, Unknotting numbers and knot diagrams with the minimum crossings,
Math. Sem. Notes Kobe Univ. 11 (1983), no. 2, 257-258.
[Bleiler 1984] S. A. Bleiler, A note on unknotting number, Math. Proc. Cambridge Philos. Soc. 96 (1984), 469-471.

(3)

G =  Then $u_\beta(G) = 0$.

In fact:  = 

Since G is a Θ-curve,

$u(G)=0 \Leftrightarrow$ G is isotopic to a plane graph.

G has a trefoil constituent knot.

Hence $u(G) \geqq 1$.

Thus, we have $u(G) = u^{\Gamma}(G) = u_{\gamma}(G) = u^{\Gamma}_{\gamma}(G) = 1$.//

2.6. The values of the unknotting numbers

Theorem 2.6.1. For $\forall$ given graph Γ and $\forall$ integer $n \geqq 1$, $\exists\, \infty$-many spatial graphs G of Γ such that

$$u_\beta(G) = u(G) = u_\gamma(G) = u^\Gamma(G) = u^\Gamma_\gamma(G) = n.$$

Infinite cyclic covering homology of a spatial graph

For a spatial graph G of Γ in $S^3=R^3 \cup \{\infty\}$ with a base T and oriented edges α_i(i=1,2,…,s) attaching to T.

Let $E(G)=cl(S^3-N(G))$ for a regular neighborhood N(G) of G in S^3.

Let χ: $H_1(E(G)) \to Z$ be the epimorphism sending the meridians of α_i (i=1,2,…,m) to $1 \in Z$.

Let $E(G)_\infty \to E(G)$ be the ∞-cyclic cover of E(G) associated with χ.

Let $\Lambda=Z[t,t^{-1}]$.

The homology $H_1(E(G)_\infty)$ is a finitely generated Λ-module which we denote by $M(G,T)_\infty$.

We take an exact sequence (over Λ)

$$\Lambda^a \to \Lambda^b \to M(G,T)_\infty \to 0,$$

where we take $a \geqq b$. A matrix $A(G,T)_\infty$ over Λ representing the homomorphism $\Lambda^a \to \Lambda^b$ is called a **presentation matrix** of the module $M(G,T)_\infty$.

For an integer $d \geqq 0$, the **d^{th} ideal** $\varepsilon_d(G,T)_\infty$ of $M(G,T)_\infty$ is the ideal generated by all the (b-d)-minors of $A(G,T)_\infty$.

The ideals $\varepsilon_d(G,T)_\infty$ (d=0,1,2,3,...) are invariants of the Λ-module $M(G,T)_\infty$.

Let (Δ_d) be the smallest principal ideal containing $\varepsilon_d(G,T)_\infty$. Then the Laurent polynomial $\Delta_d \in \Lambda$ is called the **d^{th} Alexamder polynomial** of $M(G,T)_\infty$.

If G is a knot (with T=ϕ), then $\Delta_0 \in \Lambda$ is called the **Alexander polynomial** of the knot G.

Assume that G^* is obtained from G by k crossing changes on α_i ($i=1,2,\ldots,m$). Then χ induces the epimorphism $\chi^*:H_1(E(G^*))\rightarrow Z$.

Let $m(G,T)_\infty$ and $m(G^*,T)_\infty$ be the numbers of minimal Λ-generators of the Λ-modules $M(G,T)_\infty$ and $M(G^*,T)_\infty$, respectively.

We use the following lemma:

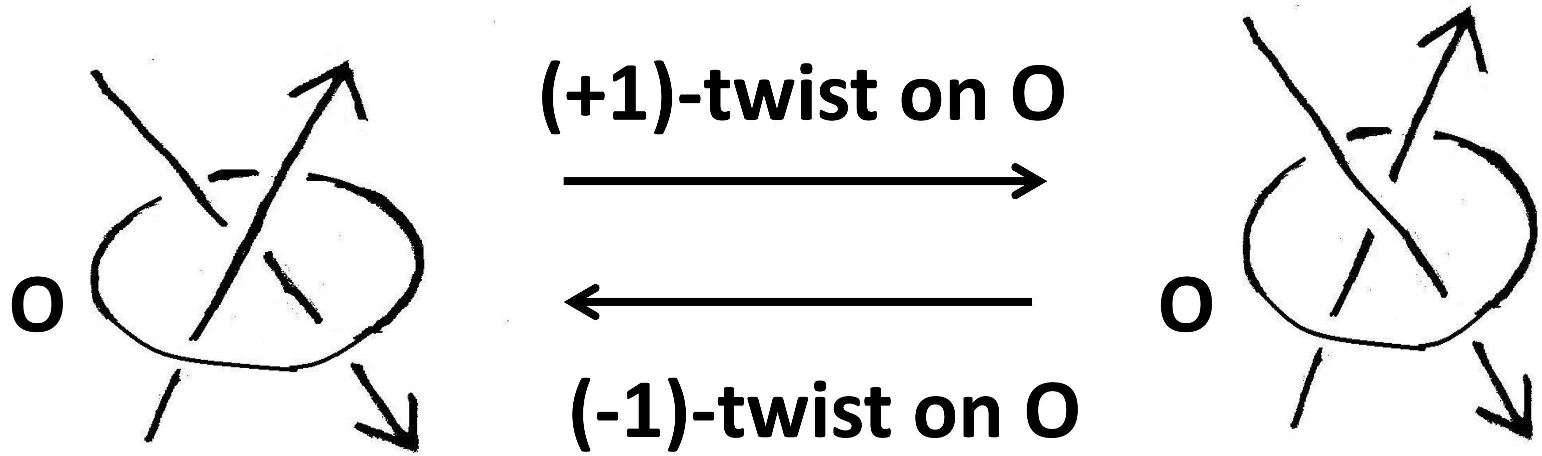
Lemma A **(cf. [Kobe J. Math. 1996]).**

$$|m(G,T)_\infty - m(G^*,T)_\infty| \leqq k.$$

[Kobe J. Math. 1996]

A. Kawauchi, Distance between links by zero-linking twists, Kobe J. Math.13(1996), 183-190.

Proof.



(-1)-crossing

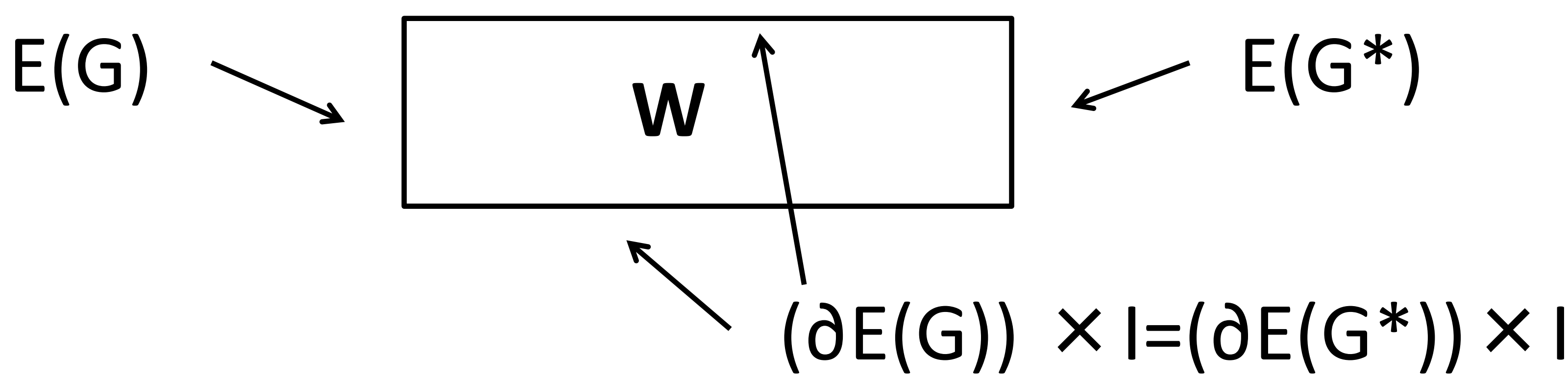
(+1)-crossing

G* is obtained from G by k crossing changes on the edges α_i (i=1,2,…,m).

G is also obtained from G* by k crossing changes on the corresponding edges α_i^* (i=1,2,…,m).

Let $W=E(G)\times I \cup_{i=1}^{n} D^2\times D^2_i$

be a surgery trace from E(G) to E(G*) by 2-handles $D^2\times D^2_i$ (i=1,2,...,n), which is also a surgery trace from E(G*) to E(G) by the “dual” 2-handles $D^2\times D^2_i$ (i=1,2,...,n).



By construction, χ and χ^* extend to an epimorphism $\chi^+ : H_1(W) \to Z$.

Let $(W_\infty; E(G)_\infty, E(G^*)_\infty)$ be the ∞-cyclic cover of $(W; E(G), E(G^*))$ associated with χ^+.

Let $m(W_\infty)$ be the minimal number of Λ-generators of the Λ-module $H_1(W_\infty)$.

Then we have

$$m(W_\infty) \leqq m(G,T)_\infty ,$$

$$m(W_\infty) \leqq m(G^*,T)_\infty .$$

Because, the natural homomorphisms

$$\pi_1(E(G)) \to \pi_1(W) \text{ and } \pi_1(E(G)) \to \pi_1(W)$$

are onto, so that the natural homomorphisms

$$H_1(E(G)_\infty) \to H_1(W_\infty) \text{ and } H_1(E(G)_\infty) \to H_1(W_\infty)$$

are onto.

By the exact sequence of the pair $(W_\infty, E(G)_\infty)$

$$H_2(W_\infty, E(G)_\infty) \to H_1(E(G)_\infty) \to H_1(W_\infty) \to 0$$

and $H_2(W_\infty, E(G)_\infty) = \Lambda^k$, we obtain

$$m(G,T)_\infty \leqq k + m(W_\infty) \leqq k + m(G^*,T)_\infty .$$

Similarly,

$$m(G^*,T)_\infty \leqq k + m(W_\infty) \leqq k + m(G,T)_\infty .$$

Thus, we have

$$|m(G,T)_\infty - m(G^*,T)_\infty| \leqq k. //$$

Proof of Theorem 2.6.1.

Let G_0 be a Γ-unknotted graph.

Let K be a trefoil knot, and K(n) the n-fold connected sum of K. Then

$$u(K(n))=u_\gamma(K(n))=n \quad \text{for } \forall n \geqq 1.$$

Let $G = G_0 \# K(n)$ be the connected sum of K(n) and an edge attaching to a base T_0 of G_0.

Then $u\Gamma_\gamma(G) \leqq n$ since $c_\gamma(G) = c_\gamma(G_0) + c_\gamma(K(n))$.

We show $u_\beta(G) \geqq n$.

Assume that $u_\beta(G)=k$. Then a β-unknotted graph G^* is obtained from G by k crossing changes on edges $\alpha_i(i=1,2,\ldots,m)$ attaching to a base T in G.

We choose orientations on α_i $(i=1,2,\ldots,m)$ as it is stated in the following two cases.

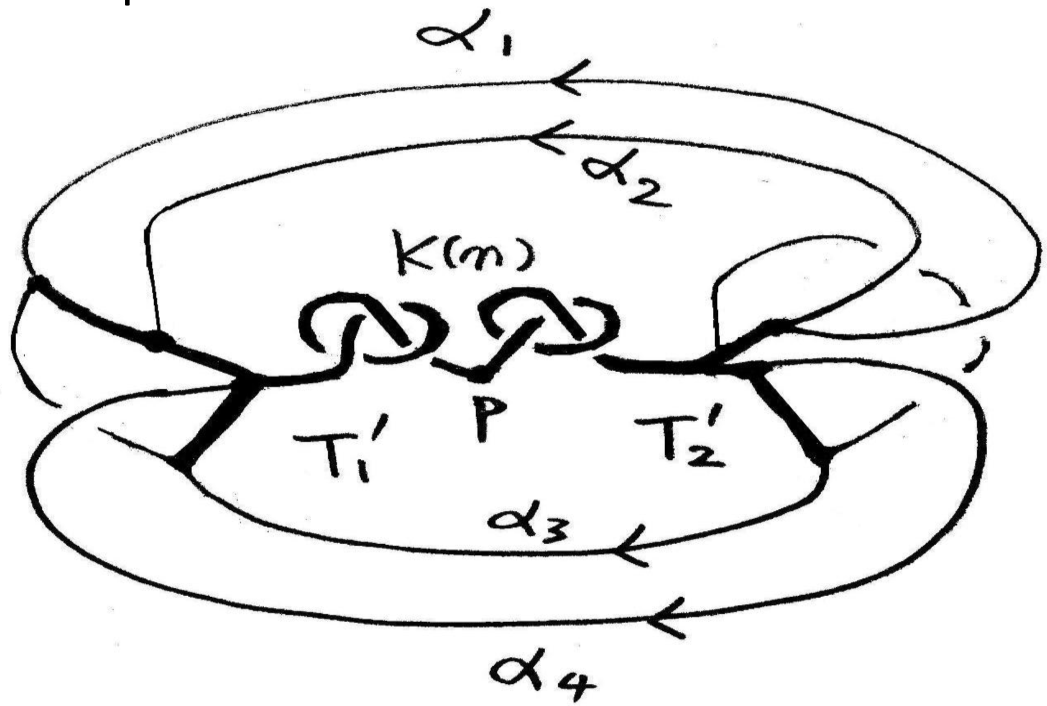
Case (I): K(n) is in an edge α_i.

Case (II): K(n) is in a component T' of the base T.

In Case (I), take any orientations on α_i $(i=1,2,\ldots,m)$.

In Case (II), let T'_1 and T'_2 be the components of $T'-\{p\}$ for a point $p \in K(n)$, and α_i (i=1,2,…,u) the edges joining T'_1 and T'_2.

We take orientations of the edges α_i (i=1,2,…,u) going from T'_2 to T'_1 and any orientations of the other edges α_i (i=u+1,u+2,…,m).



Let χ: $H_1(E(G)) \to Z$ be the epimorphism sending the oriented meridians of α_i ($i=1,2,\ldots,m$) to $1 \in Z$.

Then we have

in Case(I), $M(G,T)_\infty = \Lambda^{m-1} \oplus [\Lambda/(\Delta_K(t))]^n$, and

in Case(II), $M(G,T)_\infty = \Lambda^{m-1} \oplus [\Lambda/(\Delta_K(t^u))]^n$.

In either case, we have $m(G,T)_\infty = m+n-1$.

On the other hand, $\pi_1(E(G^*))$ is a free group of rank m and hence $M(G^*,T)_\infty = \Lambda^{m-1}$.

Thus, $m(G^*,T)_\infty = m-1$.

By Lemma A, $|(m(G,T)_\infty - m(G^*,T)_\infty| = n \leqq k$.

Hence $u_\beta(G) \geqq n$ and

$u_\beta(G) = u(G) = u_\gamma(G) = u^\Gamma(G) = u^\Gamma_\gamma(G) = n$. //

3. Applying the unknotting notions to a spatial graph attached to a surface

3.1. A spatial graph attached to a surface

Let Γ be a finite graph, and v(Γ) the set of degree one vertices. Assume $|v(\Gamma)| \geqq 2$.

Let F be a compact surface in R^3.

Definition.

A **spatial graph on F** of Γ is the image G of an embedding $f: \Gamma \to R^3$ such that

(1) G meets F with $G \cap F = f(v(\Gamma)) = v(G)$,

(2) G-v(G) is contained in one component of R^3-F,

(3) ∃ a homeomorphism $h: R^3 \to R^3$ such that $h(G \cup F)$ is a polyhedron.

- F does not need $\partial F = \phi$.
- Though Γ , G or F may be disconnected, but assume that $|F' \cap v(G)| \geqq 2$ for ∀component F’ of F.
- Ignore the degree 2 vertices in G.

<u>Definition.</u> A spatial graph G on F is **<u>equivalent</u>** to a spatial graph G' on F’ if ∃an orientation-preserving homeomorphism $h: R^3 \to R^3$ such that $h(F \cup G) = F' \cup G'$.

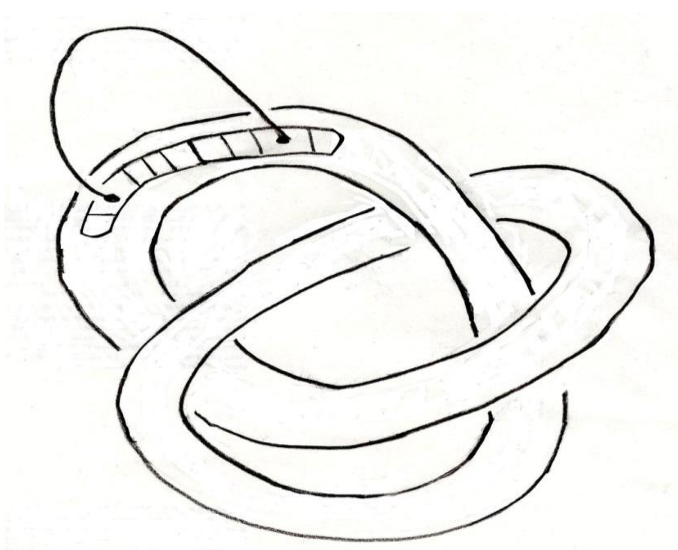
Let [G] be the class of spatial graphs G’ on F’ which are equivalent to G on F.

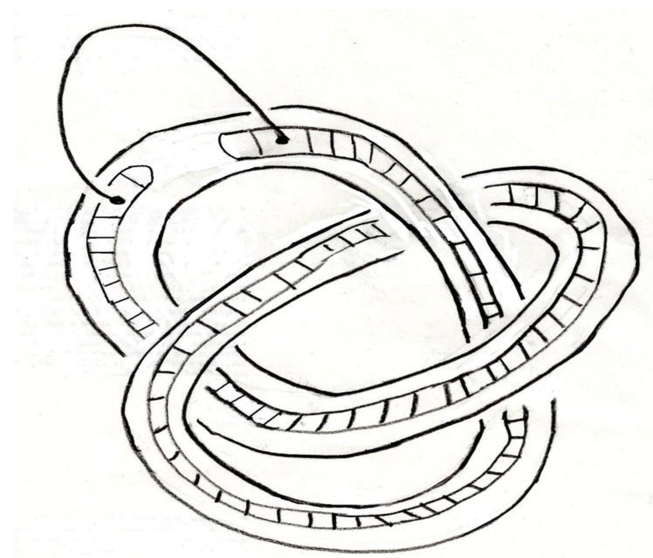
3.2. An unknotted graph on a surface and the induced unknotting number

Definition. G on F is **unknotted** if ∃ a 2-cell Δ' in ∀ component F' of F such that the union Δ of all Δ' contains v(G) and the *shrinked spatial graph* G^ with v(G^)= ϕ (i.e. a spatial graph obtained from G by shrinking ∀ Δ' into a point) is unknotted in R^3.

Note. If $\forall$ F' =S^2 or a 2-cell, then [G^] does not depend on a choice of Δ.

However, in a genral F, [G^] depends on a choice of Δ, although the *shrinked graph* Γ^ with v(Γ^)= ϕ associated with F is uniquely defined.





Because ∀G^ is a spatial graph of the same graph Γ^, we have:

Lemma. For ∀ given graph Γ and ∀ given F in R^3, ∃ only finitely many unknotted graphs G of Γ on F up to equivalences.

Let O = {unknotted graphs of Γ^}.

<u>Definition.</u>

The **<u>unknotting number</u>** u(G) of a spatial graph G of Γ on F is the distance from the set {G^} to O by crossing changes on edges attaching to a base:

$$u(G) = \rho(\{G^\wedge\}, O).$$

3.3. A β-unknotted graph on a surface and the induced unknotting number

Definition. G on F is **β-unknotted** if ∃ a 2-cell Δ' in ∀ component F' of F such that the union Δ of all Δ' contains v(G) and the shrinked spatial graph G^ with v(G^)= ϕ is β-unknotted in R^3.

unknotted ⇒ β-unknotted

Let O_β = {β-unknotted graphs of Γ^}.

Definition.

The **β-unknotting number** $u_\beta(G)$ of a spatial graph G of Γ on F is the distance from the set {G^} to O_β by crossing changes on edges attaching to a base:

$$u_\beta(G) = \rho(\{G\hat{}\}, O_\beta).$$

3.4. A γ-unknotted graph on a surface and the induced unknotting number

Definition. G on F is **γ-unknotted** if $\exists$ a 2-cell Δ' in $\forall$ component F' of F such that the union Δ of all Δ' contains v(G) and the shrinked spatial graph G^ with v(G^)= ϕ is γ-unknotted in R^3.

γ-unknotted$\Rightarrow$unknotted $\Rightarrow$ β-unknotted

Given G, let

$$\{D_{G^\wedge,\gamma}\} = \{(D;T) \in [D_{G^\wedge}] \mid c(D;T) = c_\gamma(G^\wedge),\ \forall G^\wedge\}.$$

Definition.

The **γ-unknotting number** $u_\gamma(G)$ of a spatial graph G of Γ on F is the distance from $\{D_{G^\wedge,\gamma}\}$ to O by crossing changes on edges attaching to a base:

$$u_\gamma(G) = \rho(\{D_{G^\wedge,\gamma}\}, O).$$

Note. G on F is γ-unknotted ⇔ $u_\gamma(G) = 0$.

3.5. Γ-unknotted graph on a surface and the induced unknotting numbers

Definition. G on F is Γ-**unknotted** if ∃ a 2-cell Δ' in ∀ component F' of F such that the union Δ of all Δ' contains v(G) and the shrinked spatial graph G^ with v(G^)= ϕ obtained from G by shrinking ∀Δ' into a point is Γ^-unknotted in R^3.

Γ-unknotted⇒γ-unknotted⇒unknotted

⇒ β-unknotted

Let $O_{\Gamma\wedge}$={Γ^-unknotted graphs}. Then $O_\beta \supset O \supset O_{\Gamma\wedge}$.

Definition.

The Γ-**unknotting number** $u^\Gamma(G)$ of G on F is the distance from the set {G^} to $O_{\Gamma\wedge}$ by crossing changes on edges attaching to a base:

$u^\Gamma(G) = \rho(\{G^\wedge\}, O_{\Gamma\wedge})$

The (γ,Γ)-**unknotting number** u_γ^Γ (G) of G on F is the distance from $\{D_{G\wedge,\gamma}\}$ to O_Γ by crossing changes on edges attaching to a base: $u_\gamma^G(G) = \rho(\{D_{G\wedge,\gamma}\}, O_{\Gamma\wedge})$.

3.6. Properties on the unknotting numbers

Theorem 3.6.1. The topological invariants

$$u_\beta(G),\ u(G),\ u^\Gamma(G),\ u_\gamma(G),\ u^\Gamma_\gamma(G)$$

of ∀ spatial graph G of ∀ graph Γ on ∀ surface F satisfy the following inequalities :

$$u_\beta(G) \leqq u(G) \leqq \{u_\gamma(G), u^\Gamma(G)\} \leqq u^\Gamma_\gamma(G),$$

and are distinct for some graphs G of some Γ on $F=S^2$.

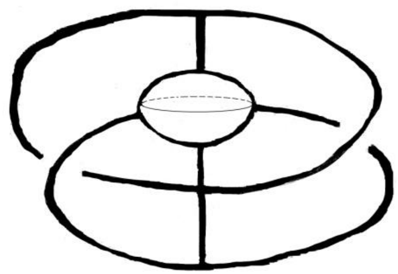
Theorem 3.6.2. For ∀ given graph Γ, ∀ surface F in R^3 and ∀ integer n ≧ 1, ∃ ∞-many spatial graphs G of Γ on F such that

$$u_\beta(G) = u(G) = u_\gamma(G) = u^\Gamma(G) = u^\Gamma_\gamma(G) = n.$$

Proof of Theorem 4.6.1. The inequalities are direct from definitions.

We show that these invariants are distinct.

(1)

G= 

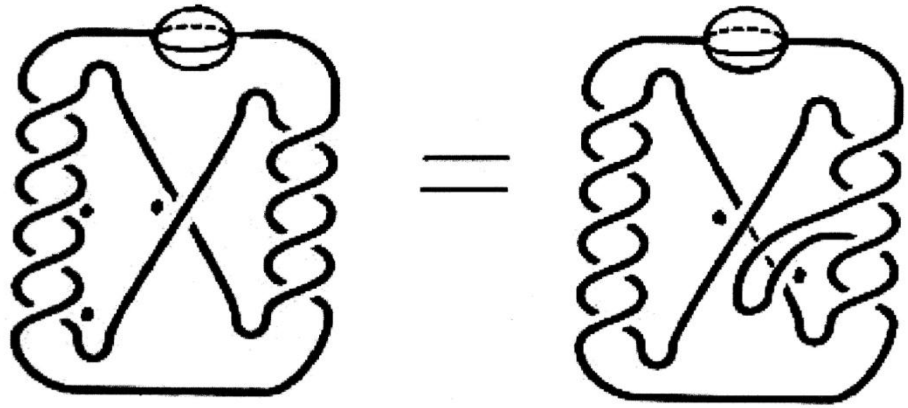
G^ has c_γ(G^)=2 and hence u_β(G)=u(G)=u_γ(G)=0.

On the other hand, we have

$$u^\Gamma(G)=u\Gamma_\gamma(G)=1,$$

for G^ is a spatial graph of a plane graph with a Hopf link as a constituent link and hence not Γ-unknotted.

(2)

G= 

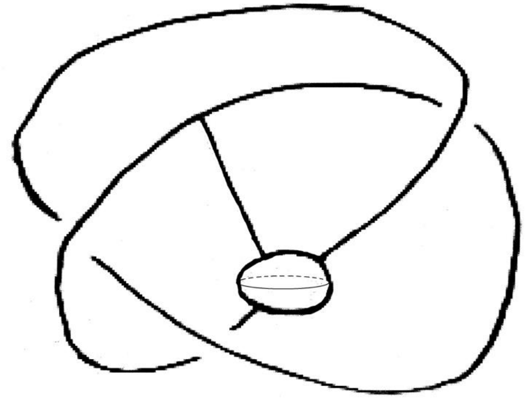
$G^{\wedge}=10_8$ has $u(10_8)=2$ and $u_\gamma(10_8)=3$

by **[Nakanishi 1983]** and **[Bleiler 1984]**.

Hence

$$u_\beta(G)= u(G) = u^\Gamma(G)=2< u_\gamma(G)=u^{\Gamma}_{\gamma}(G)=3.$$

(3)

G= 

Then $u_\beta(G)= 0$. Since G^ is a Θ-curve, by definition,

$u(G^\wedge)=0 \Leftrightarrow$ G^ is isotopic to a plane graph.

Thus, $u(G) \geqq 1$ and we have

$$u(G) = u^\Gamma(G) = u_\gamma(G) = u^\Gamma_\gamma(G) = 1.//$$

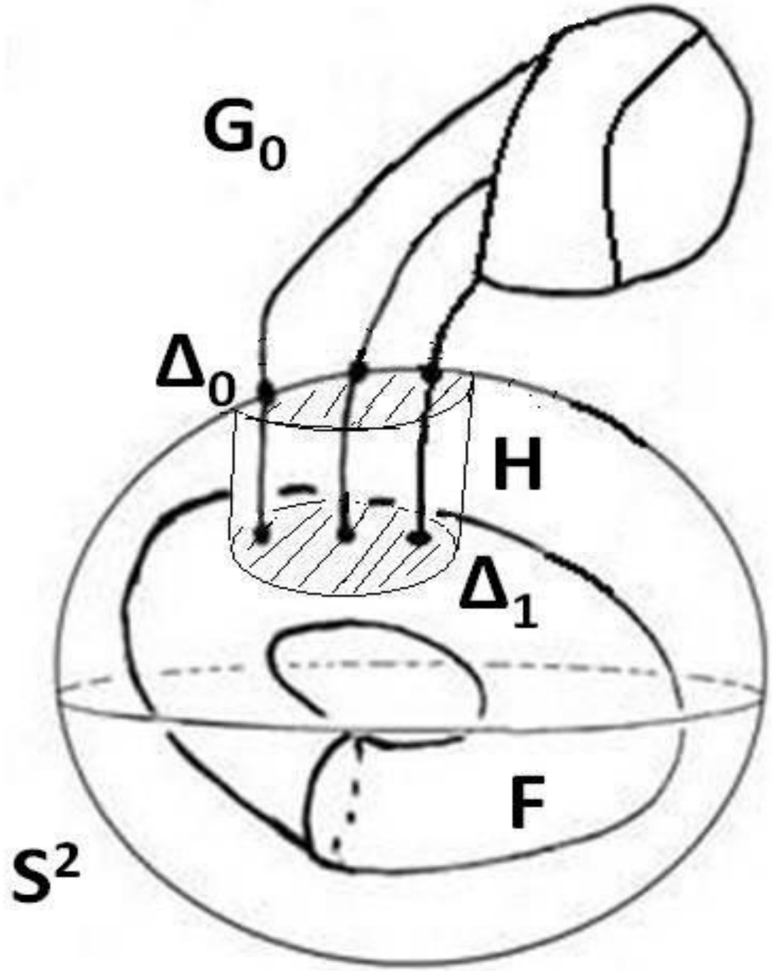
Proof of Theorem 3.6.2.

Assume $v(\Gamma) \neq \phi$.

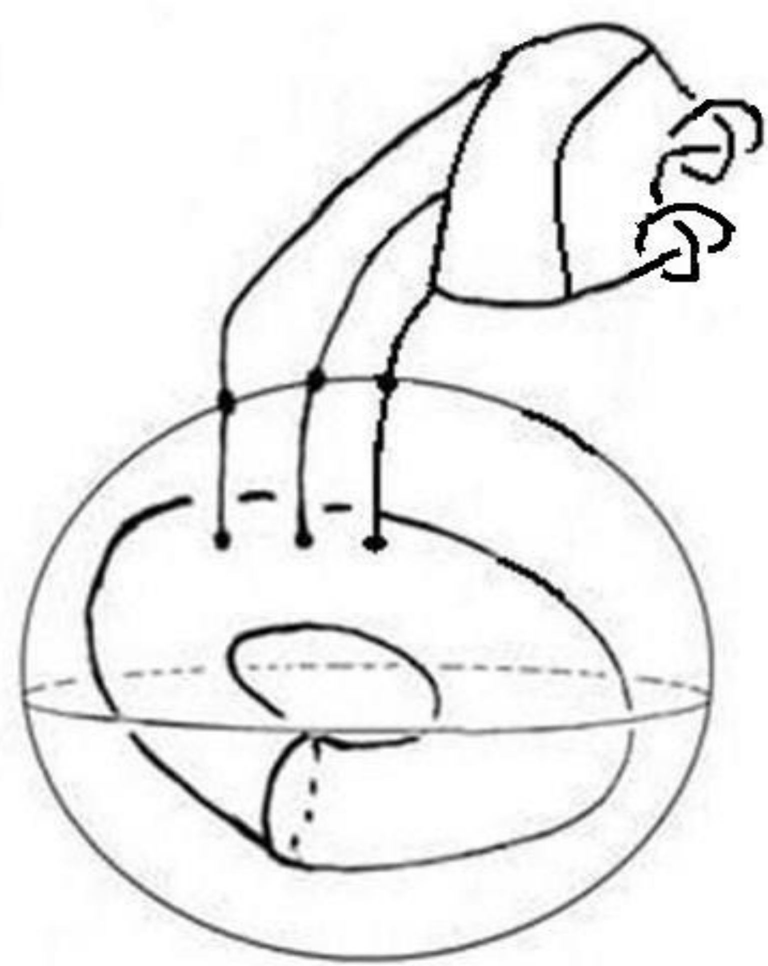
Assume Γ and F are connected for simplicity.

Let F be in the interior of a 3-ball $B \subset S^3$, and $S^2 = \partial B$.

Let G_0 be a Γ-unknotted graph on S^2 in $B^c = cl(S^3 - B)$ and extend it to a Γ-unknotted graph G_1 on F by taking in B a 1-handle H joining a 2-cell Δ_0 of S^2 and a 2-cell Δ_1 of F and then taking $|v(\Gamma)|$ parallel arcs in H.



A Γ-unknotted graph G_1 on F



A Γ-spatial graph G on F

Note that $G_0{}^\wedge = G_0 / \Delta_0$ and $G_1{}^\wedge = G_1 / \Delta_1$ are isotopic Γ-unknotted graphs in S^3.

We take a Γ-spatial graph G on F with $v(G) \subset \Delta_1$ such that $G^\wedge = G / \Delta_1$ is a connected sum $G_1{}^\wedge \# K(n)$ of an edge of $G_1{}^\wedge$ (in a part of G_0) and K(n) attaching to a base of $G_1{}^\wedge$, where K(n) is the n-fold connected sum of a trefoil knot K.

Then $u^\Gamma_\gamma(G) \leqq n$.

We show $u_{\beta}(G) \geqq n$.

Let $u_{\beta}(G) = u_{\beta}(G\text{^}')$ for $G\text{^}' = G / \Delta'$ for a 2-cell Δ' in F.

Assume that $u_{\beta}(G)=k$ and a β-unknotted graph $(G\text{^}')'$ is obtained from G^' by k crossing changes on edges α_i (i=1,2,…,m) attaching to a base T' in G^'.

As it is explained in the case $v(\Gamma) = F = \phi$, we take orientations on the edges α_i (i=1,2,…,m) and take an epimorphism $\chi: H_1(E(G\text{^}')) \to Z$.

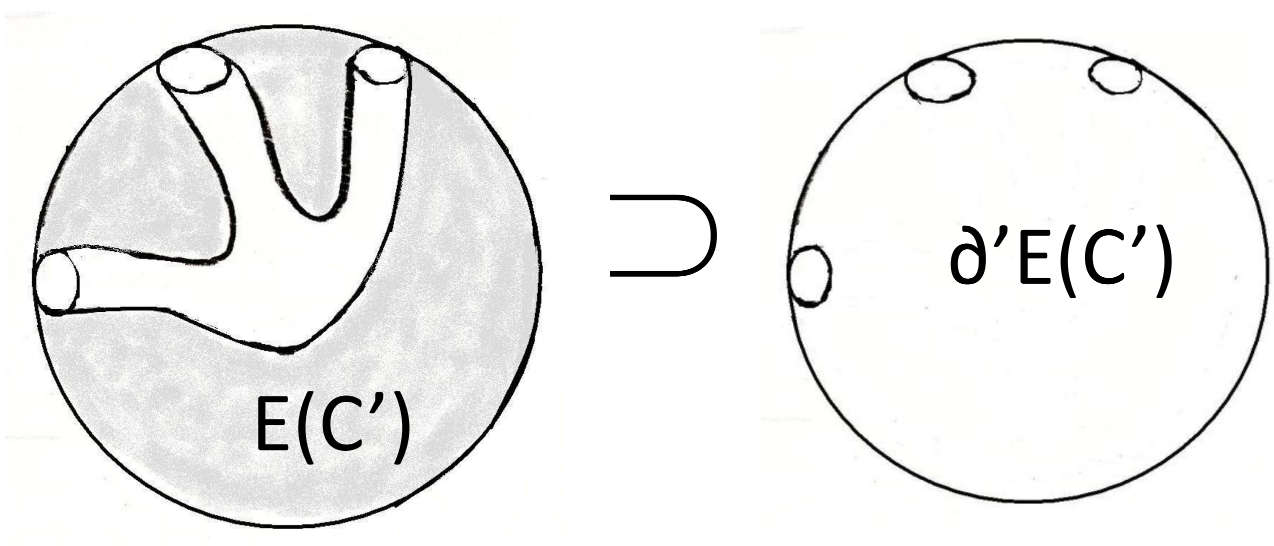
By Lemma A, $|m(G^{\wedge\prime},T')_{\infty} - m((G^{\wedge\prime})',T')_{\infty}| \leqq k$.

Note that $m((G^{\wedge\prime})',T')_{\infty} = m-1$.

Let $C' = G^{\wedge\prime} \cap B$ and $G' = G^{\wedge\prime} \cap B^{c}$. Then $G^{\wedge\prime} = G' \cup C'$.

Let $E(G') = cl(B^{c} - N(G'))$, $E(C') = cl(B - N(C'))$ and

$\partial' E(C') = E(C') \cap \partial B$.



Let $E(G')_\infty$, $E(C')_\infty$ and $\partial' E(C')_\infty$ be the lifts of $E(G')$, $E(C')$ and $\partial' E(C')$ under the covering $E(G\hat{}')_\infty \to E(G\hat{}')$, respectively.

Let

$$M(G')_\infty = H_1(E(G')_\infty) \text{ and}$$

$$M(C',\partial' C')_\infty = H_1(E(C')_\infty, \partial' E(C')_\infty).$$

Lemma B. $\exists$ a short exact sequence

$$0 \to M(G')_\infty \to M(G\hat{}', T')_\infty \to M(C', \partial' C')_\infty \to 0,$$

Further, the finite Λ-torsion part $DM(C',\partial' C')_\infty = 0$.

Proof. By excision,

$$H_d(E(G\hat{}')_\infty, E(G')_\infty) = H_d(E(C')_\infty, \partial' E(C')_\infty).$$

Since $H_d(E(C'),\partial' E(C'))=0$ for d=1,2, we see from

[Osaka J. Math. 1986]

A. Kawauchi, Three dualities on the integral homology of infinite cyclic coverings of manifolds, Osaka J. Math. 23(1986),633-651.

that $H_2(E(C')_\infty,\partial' E(C')_\infty)=0$ and $M(C',\partial' C')_\infty$ is a torsion Λ-module with $DM(C',\partial' C')_\infty=0$.

The homology exact sequence of the pair $(E(G^\wedge{}')_\infty, E(G')_\infty)$ induces an exact sequence:

$0 \to H_1(E(G')_\infty) \to H_1(E(G^\wedge{}')_\infty)$

$\to H_1(E(G^\wedge{}')_\infty, E(G')_\infty) \to 0.$

This sequence is equivalent to an exact sequence

$$0 \to M(G')_\infty \to M(G^\wedge{}', T')_\infty \to M(C', \partial' C')_\infty \to 0. \ //$$

Note that $M(G')_\infty = M(G^\wedge, T)_\infty$ for a base T of G^ corresponding to the base T'of G^'.

By an argument of the case $v(\Gamma) = F = \phi$,

$$m(G')_\infty = m(G^\wedge, T)_\infty = m+n-1$$

for the minimal number $m(G')_\infty$ of Λ-generators of $M(G')_\infty$.

Lemma C **(cf. [Kobe J. Math,1987]).**
Let M' be a Λ-submodule of a finitely generated Λ-module M. Let m' and m be the minimal numbers of Λ-generators of M' and M, respectively. If D(M/M') =0, then m' $\leqq$ m.

[Kobe J. Math,1987]

A. Kawauchi, On the integral homology of infinite cyclic coverings of links, Kobe J. Math. 4(1987),31-41.

Proof. For a Λ-epimorphism f: $\Lambda^m \to M$, let $B=f^{-1}(M')\subset\Lambda^m$, which is mapped onto M'. Since Λ^m/B is isomorphic to M/M' which has projective dimension $\leqq 1$, B is Λ-free, i.e., $B=\Lambda^b$ with $b \leqq m$. Hence $m' \leqq b \leqq m$.//

By Lemma C,

$$m(G^{\wedge\prime},T')_{\infty} \geqq m(G')_{\infty} = m+n-1.$$

Since $m((G^{\wedge\prime})',T')_{\infty} = m-1$, we have

$$k \geqq m(G^{\wedge\prime},T')_{\infty} - m((G^{\wedge\prime})',T')_{\infty} \geqq n.$$

Hence $u_{\beta}(G) \geqq n$ and

$$u_{\beta}(G)= u(G)= u_{\gamma}(G)= u^{\Gamma}(G)= u^{\Gamma}_{\gamma}(G)=n.//$$

Thank you for your attention.
ご静聴ありがとうございました。