NEWLY FOUND FORBIDDEN GRAPHS FOR TRIVIALIZABILITY

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ABSTRACT

A planar graph is said to be trivializable if every regular projection produces a trivial embedding for some over/under informations. Every minor of a trivializable graph is also trivializable, thus the set of forbidden graphs is finite. Seven forbidden graphs for the trivializability were previously known. In this paper, we exhibit nine more forbidden graphs.

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1. Introduction

Let G be a finite graph. We consider G as a topological space in the usual way. An embedding of G into $\mathbb{R}^3$ or S^3 is called a *spatial embedding* of G or simply a *spatial graph*. A graph G is said to be *planar* if there exists an embedding of G

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into $\mathbb{R}^2$. A spatial embedding of a planar graph is said to be *trivial* if it is ambient isotopic to an embedding into $\mathbb{R}^2 \subset \mathbb{R}^3$.

A *regular projection* of G is a continuous map $\hat{f} : G \to \mathbb{R}^2$ whose multiple points are only finitely many transversal double points away from the vertices of G. In this paper $\hat{G}$ denotes the image of a regular projection of G. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the natural projection. Then, for a regular projection $\hat{f}$ of G, if we give over/under information to each double point then the regular projection represents a spatial embedding $f : G \to \mathbb{R}^3$ such that $\hat{f} = \pi \circ f$. Then we say that f *is obtained from* $\hat{f}$ and also call $\hat{f}$ a *regular projection of* f. A regular projection $\hat{f}$ of a planar graph G is called a *knotted projection* [7] if all spatial embeddings of G which can be obtained from $\hat{f}$ are non-trivial. For example, the graph G_1 illustrated in Fig. 1-(1) has a knotted projection $\hat{G}_1$ as in Fig. 2-(1) which always yields at least one Hopf link. This was pointed out by the third author [7].

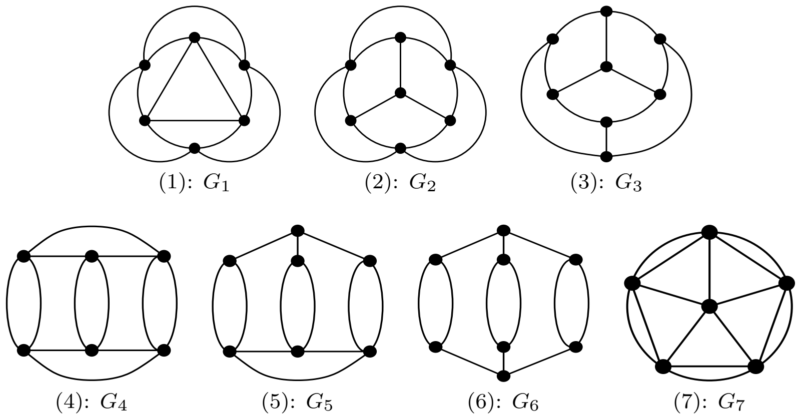


Fig. 1. Previously known seven elements of $\Omega(\mathcal{T})$.

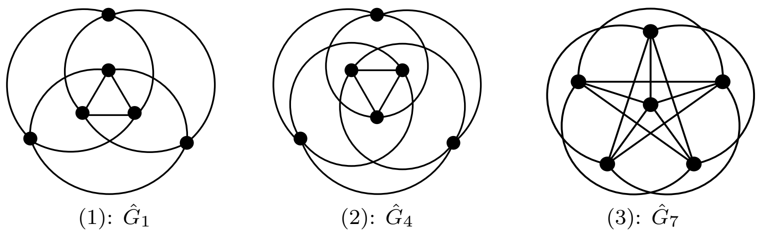


Fig. 2. Knotted projections of G_1, G_4 and G_7.

A planar graph is said to be *trivializable* if it has no knotted projections. For example, if G is homeomorphic to the disjoint union of 1-spheres then G is trivializable. We remark here that there exist infinitely many trivializable graphs as follows. The third author showed that every *bifocal* illustrated in Fig. 3-(1) is trivializable [7]. In [6], N. Tamura showed that every *neobifocal* is trivializable. I. Sugiura and

S. Suzuki showed that any *three-line web* is trivializable [5], where a three-line web is a graph obtained from six trees and cycles that contains exactly three edges as follows. Let T_1^+, T_1^-, T_2^+, T_2^-, T_3^+ and T_3^- be disjoint six trees. Let C_i be a cycle which consists of three vertices v_1^i, v_2^i and v_3^i and three edges $v_2^i v_3^i$, $v_3^i v_1^i$ and $v_1^i v_2^i$ for $i = 1, \ldots, n$. We join v_j^i and v_j^{i+1} by an edge for $i = 1, \ldots, n$ and $j = 1, 2, 3$. Let u^+ and u^- be vertices not on these trees and cycles. We join u^+ (resp. u^-) and each vertex of T_i^+ (resp. T_i^-) by an edge for $i = 1, 2, 3$. Let w_i^+ (resp. w_i^-) be a vertex of T_i^+ (resp. T_i^-) for $i = 1, 2, 3$. Finally we identify w_i^+ with v_i^1 and w_i^- with v_i^n for $i = 1, 2, 3$ and obtain a three-line web as in Fig. 3-(2).

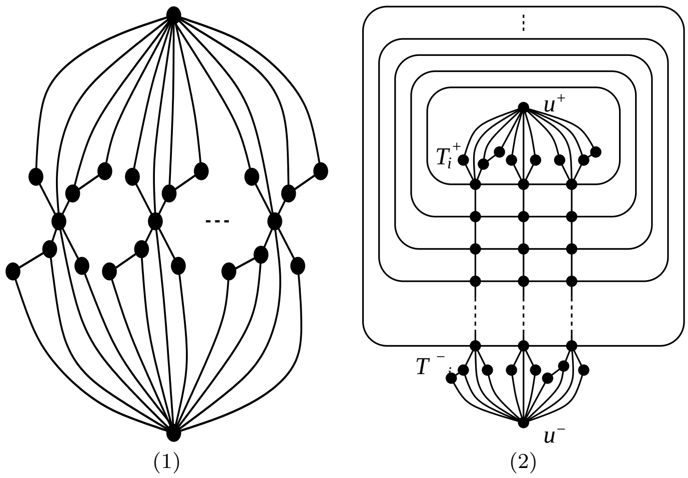


Fig. 3. A bi-focal and a three-line web.

A *minor* of a graph G is a graph which is obtained from G by a finite sequence of an edge contraction or taking a subgraph. It is known that the trivializability is inherited by minors [7, Proposition 1.1]. Let $\Omega(\mathcal{T})$ be the set of non-trivializable graphs whose all proper minors are trivializable. This is called the *obstruction set* for the trivializability and each of elements in $\Omega(\mathcal{T})$ is called a *forbidden graph* for the trivializability. Then, according to Robertson-Seymour [2], $\Omega(\mathcal{T})$ is a finite set.

Problem 1.1. *Find all forbidden graphs for the trivializability.*

In [5], Sugiura and Suzuki showed that the seven graphs listed in Fig. 1 belong to $\Omega(\mathcal{T})$. They showed that each of $G_2, G_3, \ldots, G_7$ has a knotted projection as illustrated in Fig. 2 and each minor of these seven graphs is a minor of a three-line web. In this paper, we exhibit twelve graphs each of which has a knotted projection and show that nine of them are new forbidden graphs for the trivializability.

Theorem 1.2. *Each of the graphs G_i $(i = 8, 9, \ldots, 16)$ illustrated in Fig. 4 and H_i $(i = 1, 2, 3)$ illustrated in Fig. 6-(1.1), (2.1) and (3.1) has a knotted projection. Moreover, each of the graphs G_i $(i = 8, 9, \ldots, 16)$ belongs to $\Omega(\mathcal{T})$.*

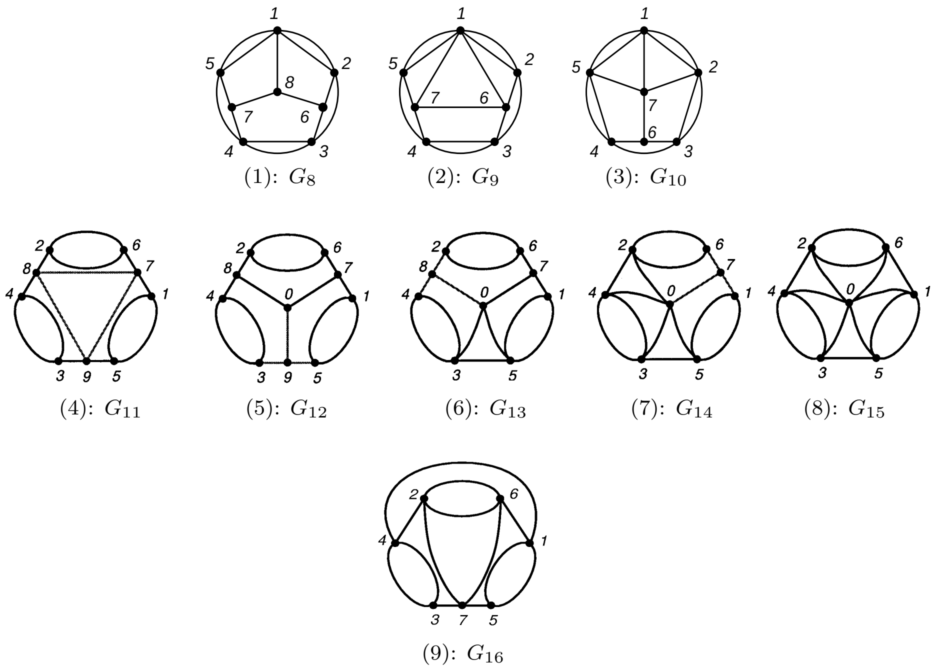


Fig. 4. Newly found forbidden graphs G_8, ..., G_{16}.

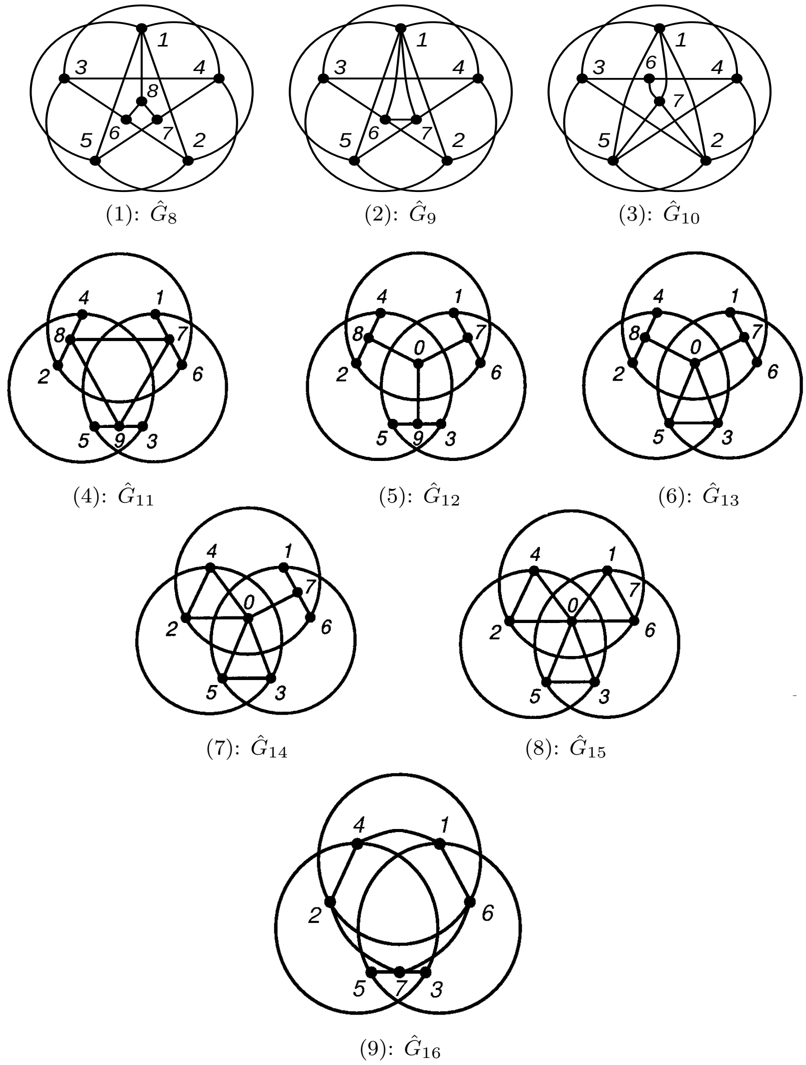
Then, at this writing, we have sixteen elements of $\Omega(\mathcal{T})$. In Fig. 5 we illustrate knotted projections of these graphs. It is not hard to observe the following:

Proposition 1.3. *Each spatial embedding obtained from the regular projections illustrated in Fig. 5 and Fig. 6-(1.2), (2.2) and (3,2) contains a Hopf link.*

We note that every known knotted projection has this property. Notice that each of H_1, H_2 and H_3 contains the *eye-graph* illustrated in Fig. 7 as a minor. Sugiura asked in his master thesis [4] whether or not this graph is trivializable. As far as the authors know, it is still open. We observe that a spatial embedding which has no non-trivial 2-component link can be obtained from every regular projection of the eye-graph (Proposition 3.1). Then, we ask the following:

Question 1.4. Does any spatial embedding which is obtained from knotted projections contain a non-trivial link?

A spatial embedding of a graph is said to be *free* if the fundamental group of the spatial graph-exterior is a free group. In section 3 we show that a free spatial embedding can be obtained from any regular projection of an arbitrary graph (Proposition 3.2). Besides we mention the topics related to regular projections of *minimally knotted* spatial embeddings of a planar graph.



(1): $\hat{G}_8$ (2): $\hat{G}_9$ (3): $\hat{G}_{10}$

(4): $\hat{G}_{11}$ (5): $\hat{G}_{12}$ (6): $\hat{G}_{13}$

(7): $\hat{G}_{14}$ (8): $\hat{G}_{15}$

(9): $\hat{G}_{16}$

Fig. 5. Knotted projections of $G_8, \ldots, G_{16}$.

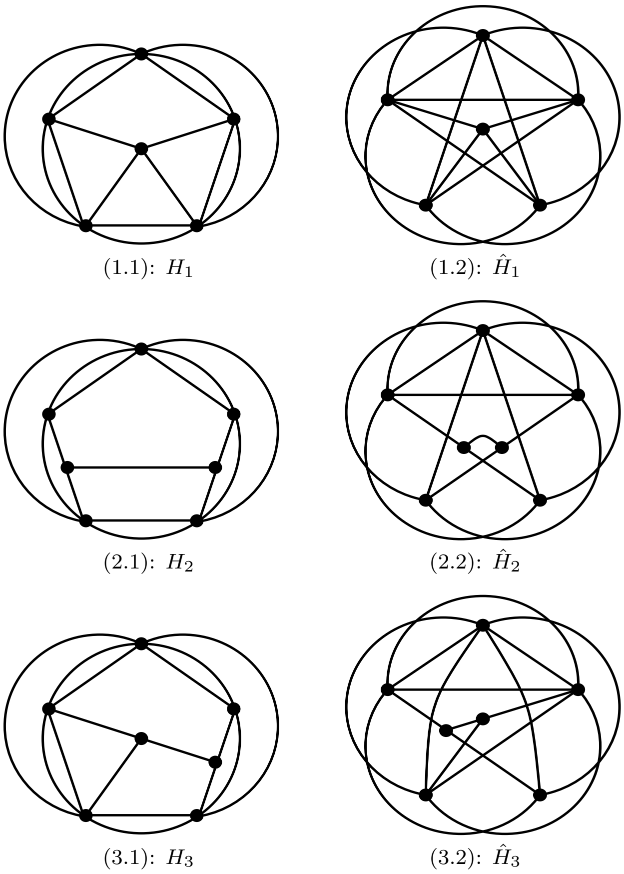


Fig. 6. A major H_i of the eye-graph and its knotted projection $\hat{H}_i$.

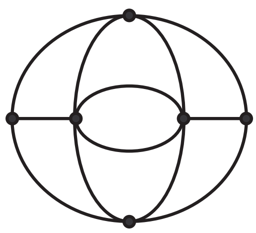


Fig. 7. The eye-graph.

2. Proof of Theorem 1.2

Proof of Proposition 1.3. It is direct to see that $\hat{G}_i$ contains a projection $\hat{H}$ of the $(2, 5)$-torus knot if $i = 8, 9, 10$ and the Borromean rings if $i = 11, 12, \ldots, 16$. For any over/under information of $\hat{H}$, there is an "alternating part" of crossings. It is sufficient to deal with the three cases as in Fig. 8. In each case, we can find a Hopf link in the spatial embedding obtained from the projection other than the specified two double points. This completes the proof. □

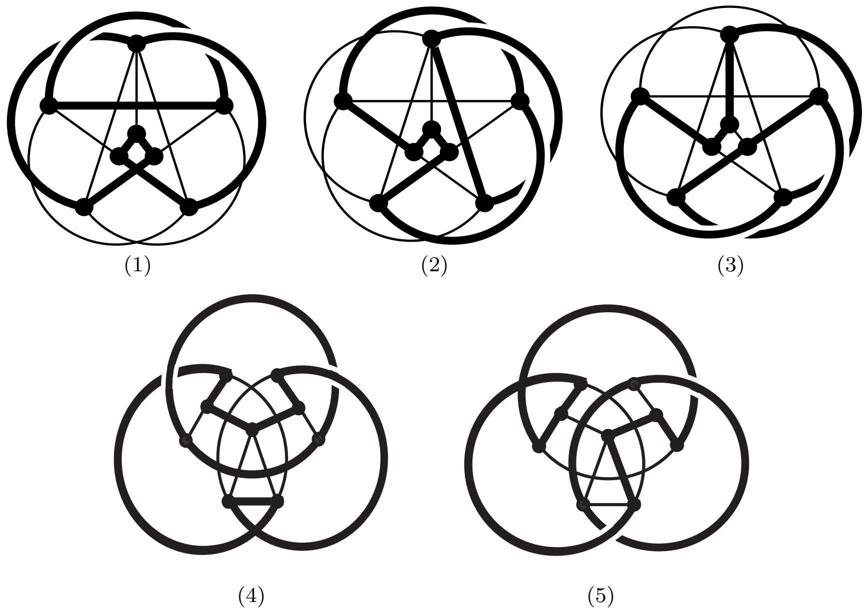


Fig. 8. These projections always yield the Hopf link.

Proof of Theorem 1.2. By Proposition 1.3, we see that G_i $(i = 8, 9, \ldots, 16)$ has a knotted projection. We show that each proper minor of G_i $(i = 8, 9, \ldots, 16)$ is trivializable. Here we recall that for two trivializable graphs G, G' and vertices v, v' of G, G' respectively, the graph obtained from G and G' by identifying $v = v'$ is trivializable [7, Proposition 3.2 (2)]. Then we see from Fig. 9, 10, …, 17 and from this fact that each proper minor of G_i $(i = 8, 9, \ldots, 16)$ is a minor of a three-line web. This completes the proof. □

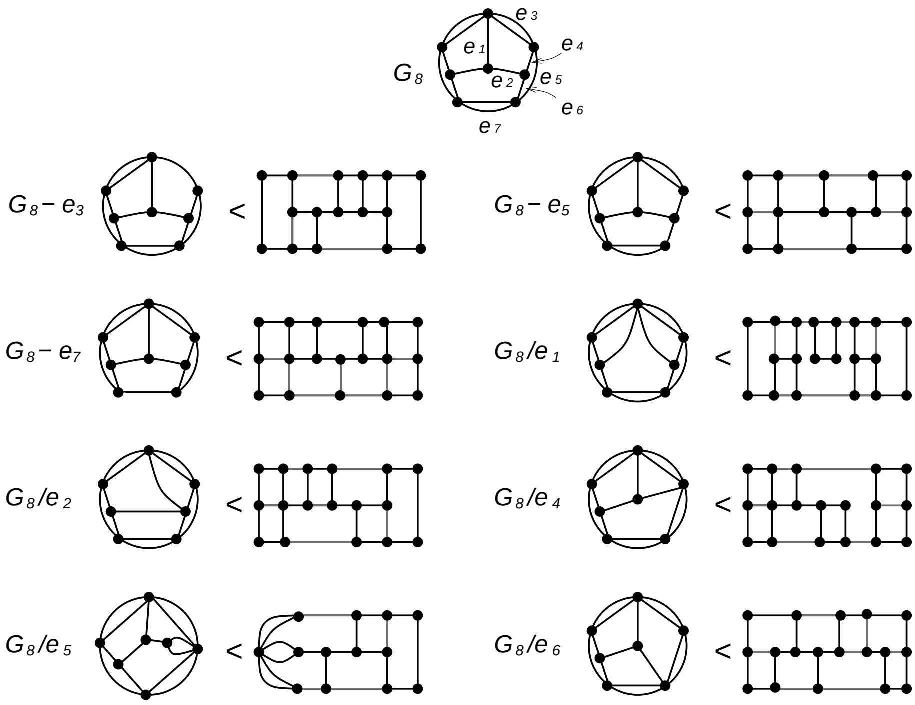


Fig. 9. $G_8 - e_i$, G_8/e_i.

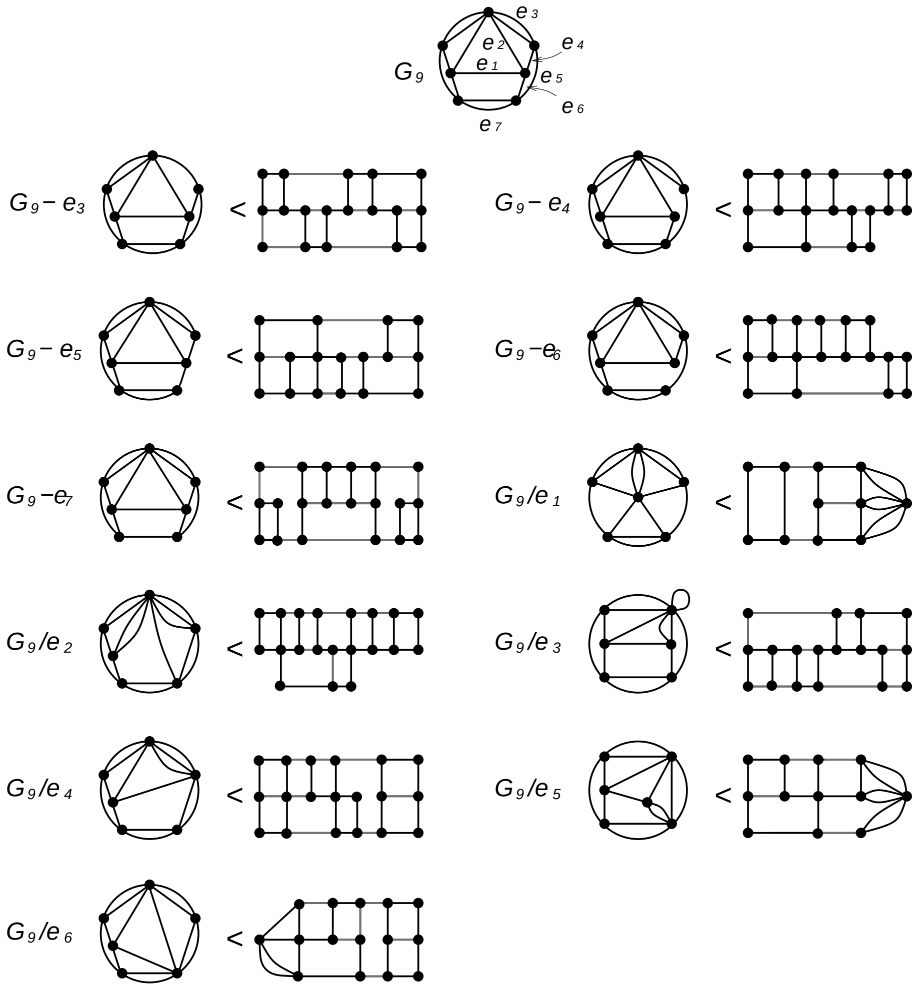


Fig. 10. $G_9 - e_i$, G_9/e_i.

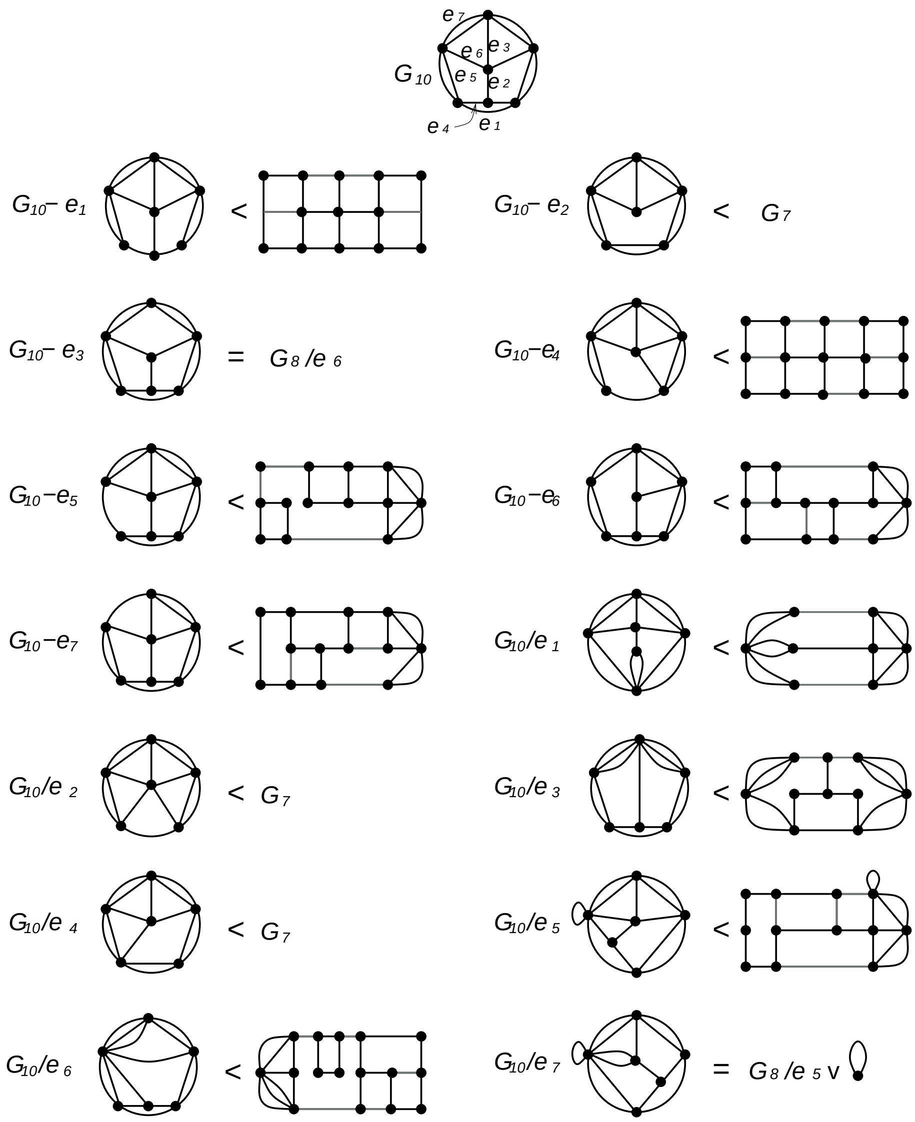


Fig. 11. $G_{10} - e_i$, G_{10}/e_i.

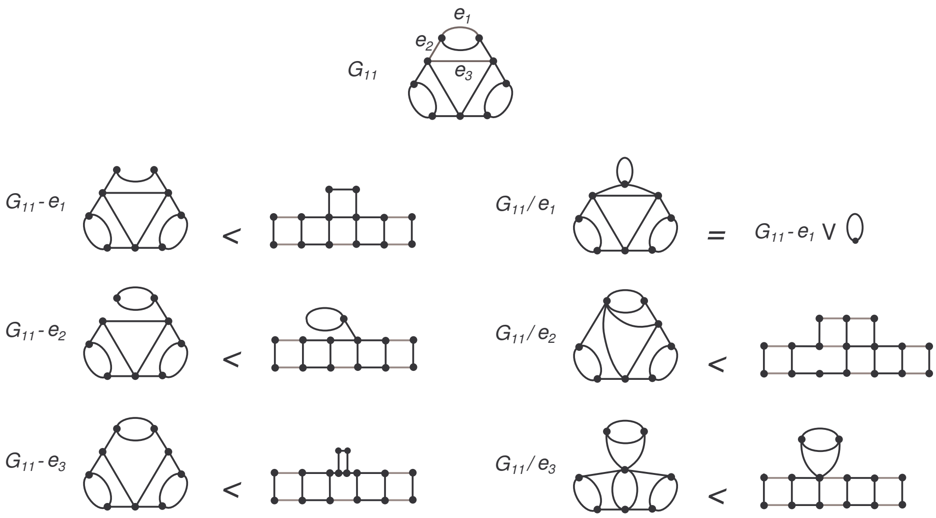


Fig. 12. $G_{11} - e_i$, G_{11}/e_i.

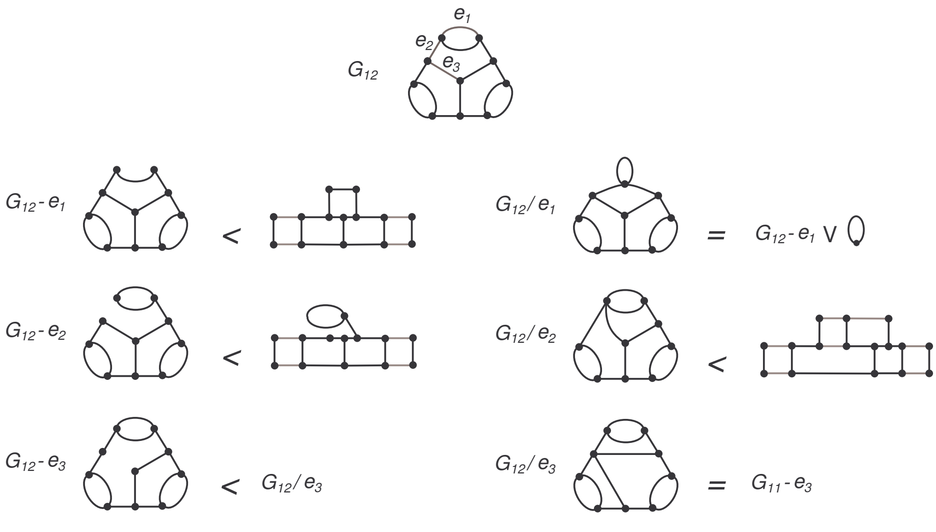


Fig. 13. $G_{12} - e_i$, G_{12}/e_i.

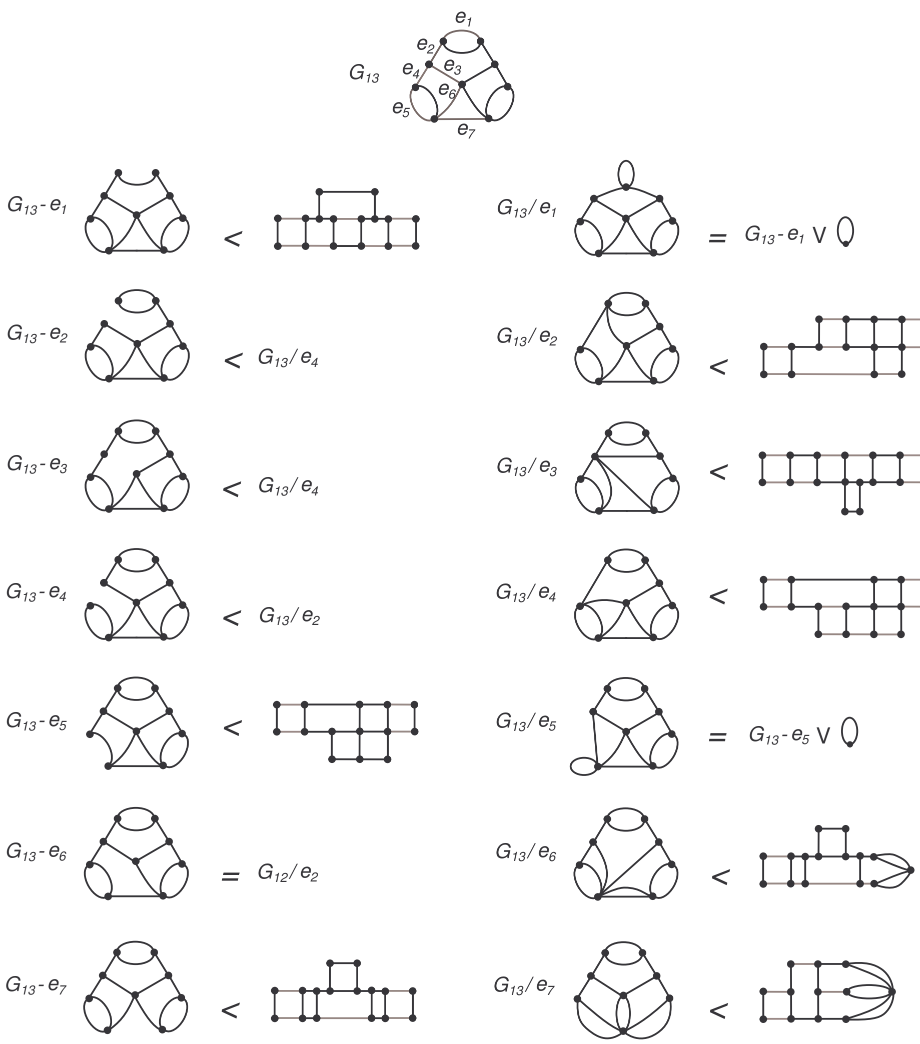


Fig. 14. $G_{13} - e_i$, G_{13}/e_i.

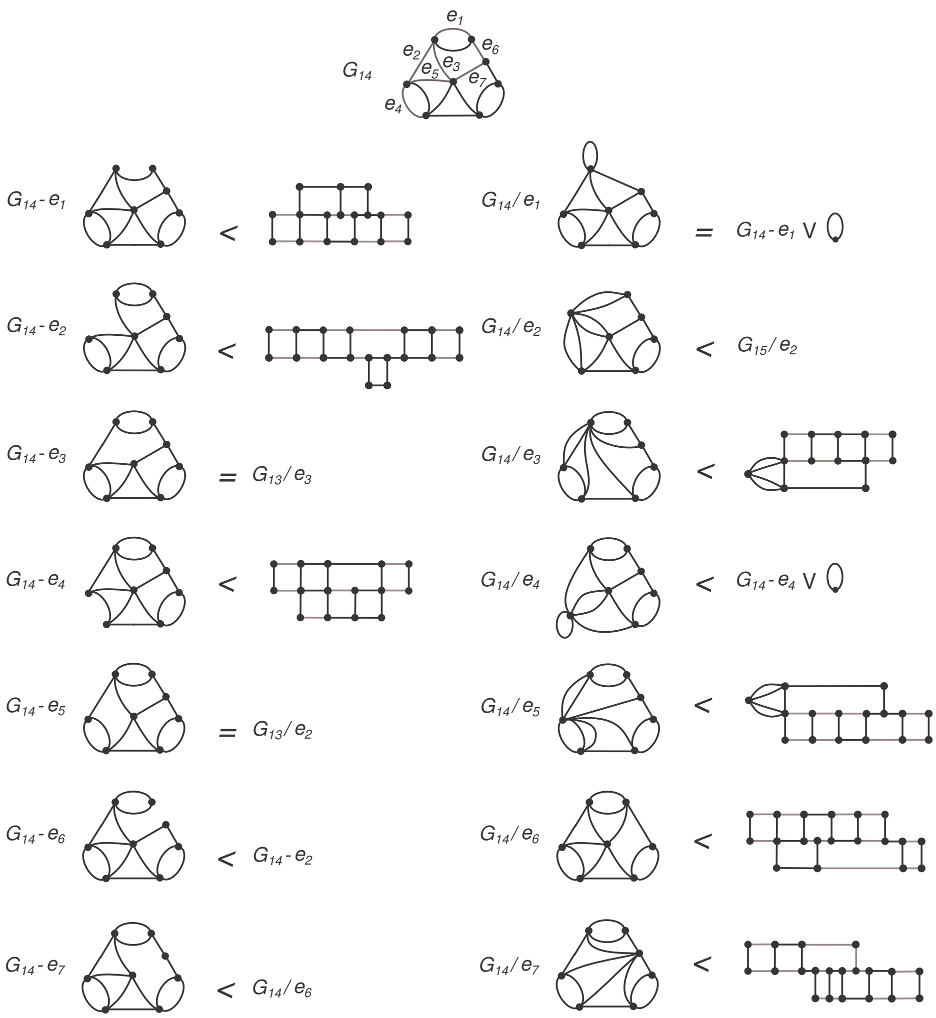


Fig. 15. $G_{14} - e_i$, G_{14}/e_i.

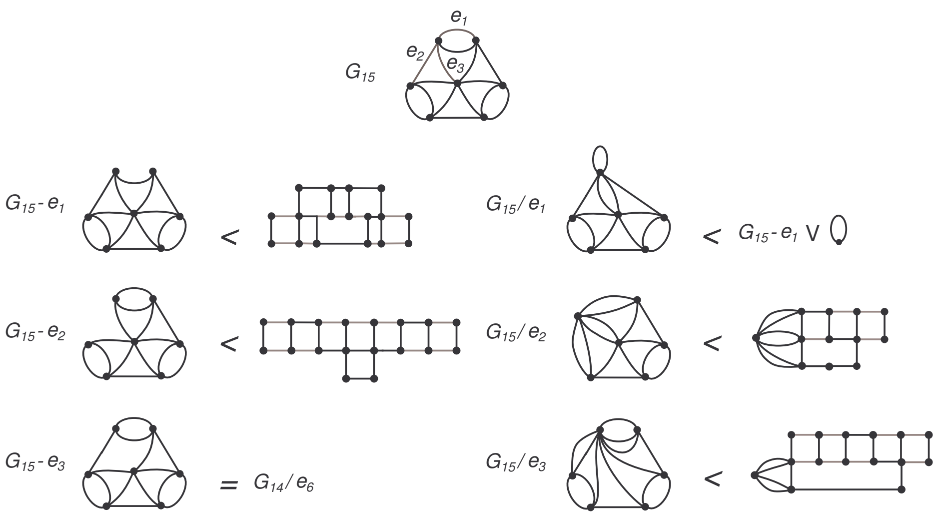


Fig. 16. $G_{15} - e_i$, G_{15}/e_i.

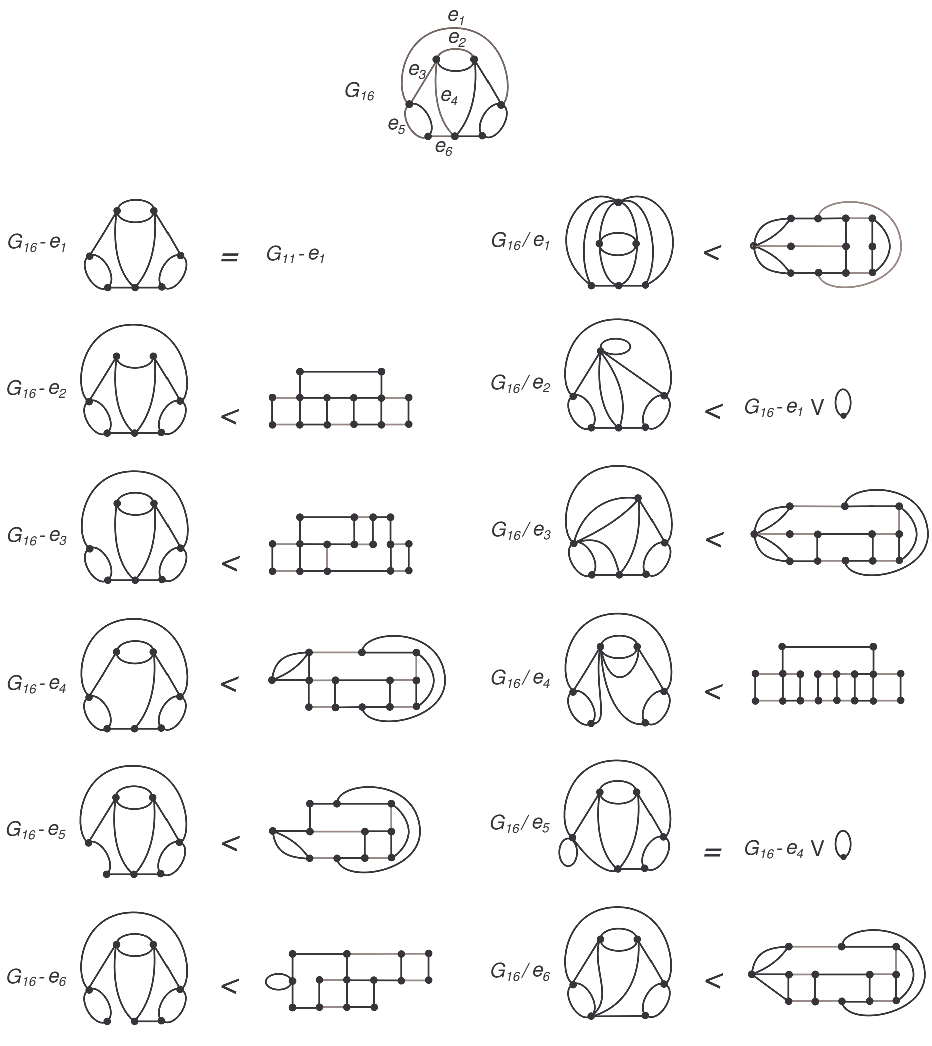


Fig. 17. $G_{16} - e_i$, G_{16}/e_i.

3. Remarks

Given labels of vertices and edges of a graph, we give over/under information to regular projections in the following manner: (1) For a double point of distinct edges, the edge with the smaller label is over. (2) For a double point of the same edges, the one near the vertex with the smaller label is over.

Proposition 3.1. *The spatial embedding obtained from any regular projection of the eye-graph under the over/under informations defined by Fig. 18 contains no non-trivial links.*

Proof. We give the labels of vertices and edges of the eye-graph as illustrated in Fig. 18. In the eye graph there exist exactly three pairs $(\gamma(1,4,5),\gamma(6,9,12))$, $(\gamma(2,3,4),\gamma(7,9,11))$, $(\gamma(1,2,6,7),\gamma(8,10))$ of disjoint cycles, where γ denotes the cycle uniquely determined by the specified labels of edges. It is easy to check that each of the six cycles appeared in these pairs forms a trivial knot, and each corresponding 2-component link is split. This completes the proof. □

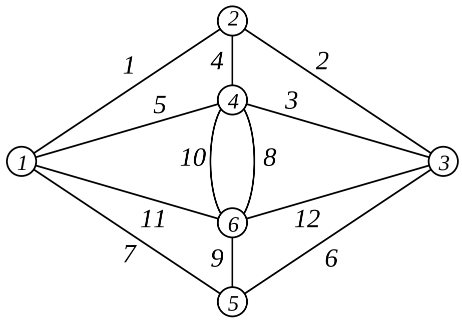


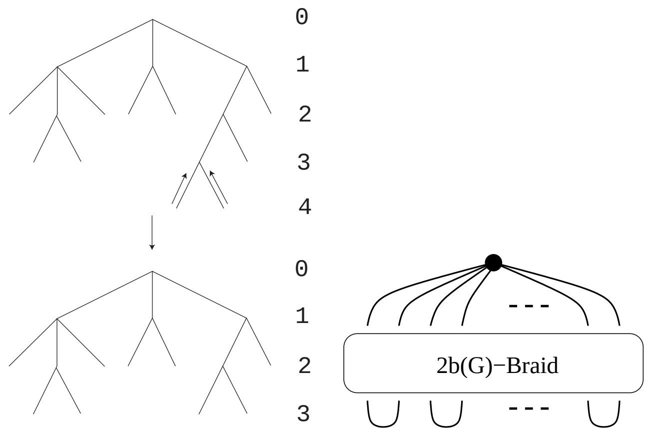
Fig. 18.

We conclude this paper with the following proposition and related problems.

Proposition 3.2. *Let G be a finite graph and $\hat{f}$ a regular projection of G. Then there exists a free spatial embedding can be obtained from $\hat{f}$.*

We note that G does not need to be planar in Proposition 3.2.

Proof of Proposition 3.2. We may assume that G is connected. Let $\hat{f}$ be a regular projection of a graph G. Let T be a spanning tree of G. We fix a vertex v_0 of G and give over/under information to the regular projection regarding a usual metric on T to produce a spatial embedding $f : G \to \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ such that, where p_1 (resp. p_2) denotes the projection of $\mathbb{R}^3$ to the first factor (resp. to the second factor) of $\mathbb{R}^2 \times \mathbb{R}$, $\hat{f} = p_1 \circ f$ and $p_2 \circ f|_T$ forms a height function coinciding with the metric on T as Figure 19-(1) which extends to G as a continuous map $h : f(G) \to \mathbb{R}$ so that each spatial edge of $f(G - T)$ has exactly one minimum point.



(1) (2): $f(G)/f(T)$

Fig. 19.

Then by contracting spatial edges of $f(T)$, we obtain a bouquet as in Fig. 19-(2) which is trivial. Now we see that $\pi_1(\mathbb{R}^3 - f(G))$ is a free group. This completes the proof of Proposition 3.2. $\square$

A spatial embedding of a planar graph G is said to be *minimally knotted* if it is non-trivial but $f|_H$ is trivial for any proper subgraph H of G. A spatial embedding $f : G \to S^3$ is said to be *totally knotted* if the natural homomorphism $i_* : \pi_1(\partial E(f(G))) \to \pi_1(E(f(G)))$ is injective, where $E(f(G))$ denotes the spatial graph-exterior. The second and the last authors showed the following in [1].

Theorem 3.3. *Any minimally knotted spatial embedding of a planar graph is totally knotted.*

We call a regular projection of a planar graph G a *minimally knotted projection* if all spatial embeddings of G obtained from it are minimally knotted.

Proposition 3.4. *Any planar graph does not have a minimally knotted projection.*

Proof. Suppose that there exists a planar graph G which has a minimally knotted projection $\hat{f}$ of G. By Proposition 3.2, there exists a spatial embedding f of G obtained from $\hat{f}$ such that the fundamental group of the exterior of $f(G)$ is a free group. On the other hand, by Theorem 3.3 we have that f is totally knotted. Thus, the fundamental group of the exterior of $f(G)$ is not a free group. This is a contradiction. $\square$

Remark 3.5. We can also prove Proposition 3.4 by using Scharlemann-Thompson's result [3] instead of Theorem 3.3.

Proposition 3.6. *Let $\hat{f}$ be a knotted projection of a planar graph G and f arbitrary spatial embedding of G obtained from $\hat{f}$. Then there exists a subgraph H of G such that $f|_H$ is a non-free spatial embedding.*

Proof. By Proposition 3.2, there exists a free spatial embedding f obtained from $\hat{f}$. By Scharlemann-Thompson [3], there exists a subgraph H of G such that $f|_H$ is a non-free spatial embedding since f is not trivial. □

Then, it is natural to ask the following.

Question 3.7. Is any regular projection of a minimally knotted spatial embedding of a planar graph not a knotted projection?

It can be easily seen that Question 3.7 is equivalent to the following question.

Question 3.8. Let $\hat{f}$ be a knotted projection of a planar graph G and f arbitrary spatial embedding of G obtained from $\hat{f}$. Does there exist a 'proper' subgraph H of G such that $f|_H$ is a non-free spatial embedding?

References

[1] M. Ozawa and Y. Tsutsumi, *Minimally knotted spatial graphs are totally knotted*, Tokyo J. Math. **26** (2003) 413–421.
[2] N. Robertson and P. Seymour, *Graph minors XVI. Wagner's conjecture*, to appear.
[3] M. Scharlemann and A. Thompson, *Detecting unknotted graphs in* 3*-space*, J. Diff. Geom. **34** (1991) 539–560.
[4] I. Sugiura, *On trivializability of spatial graph projections*, Master thesis, Waseda University (1997).
[5] I. Sugiura and S. Suzuki, *On a class of trivializable graphs*, Sci. Math. **3** (2000) 193–200.
[6] N. Tamura, *On an extension of trivializable graphs*, J. Knot Theory Ramifications **13** (2004) 211–218.
[7] K. Taniyama, *Knotted projections of planar graphs*, Proc. Amer. Math. Soc. **123** (1995) 3575–3579.