On the knotted projections of spatial graphs*

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1. Knotted projections of spatial graphs

Let G be a finite graph. We give a label to each of vertices and edges of G. An embedding of G into $\mathbf{R}^3$ is called a *spatial embedding* of G or simply a *spatial graph*. A graph G is said to be *planar* if there exists an embedding of G into $\mathbf{R}^2$. A spatial embedding of a planar graph G is said to be *trivial* if it is ambient isotopic to an embedding of G into $\mathbf{R}^2 \subset \mathbf{R}^3$. We note that a trivial spatial embedding of a planar graph is unique up to ambient isotopy in $\mathbf{R}^3$ [2].

A *regular projection* of G is an immersion $G \to \mathbf{R}^2$ whose multiple points are only finitely many transversal double points away from vertices. Let $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ be the natural projection. For a regular projection $\hat{f}$ of G, by giving over/under information to each double point, we can obtain a regular diagram of a spatial embedding f of G such that $\hat{f} = \pi \circ f$. Then we say that f is *obtained from* $\hat{f}$.

It is well known that for any regular projection $\hat{f}$ of G which is homeomorphic to the disjoint union of 1-spheres there exists a trivial spatial embedding of G which is obtained from $\hat{f}$. This fact plays an important role in knot theory. For example, the theory of skein polynomial invariants is based on this fact. But this fact does not always hold for a regular projection of a planar graph. Let G be the octahedron graph and $\hat{f}$ a regular projection of G as illustrated in Fig.

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1.1. K. Taniyama pointed out that any of the spatial embeddings of G which is obtained from $\hat{f}$ is non-trivial [11]. A regular projection $\hat{f}$ of a planar graph G is said to be *knotted* if any spatial embedding of G which is obtained from $\hat{f}$ is non-trivial.

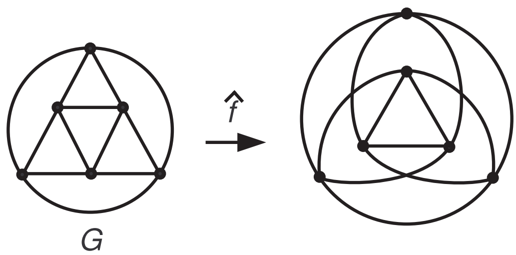

Fig. 1.1. A knotted projection $\hat{f}: G \to \mathbf{R}^2$

A planar graph is said to be *trivializable* if it has no knotted projections. Thus a graph which is homeomorphic to the disjoint union of 1-spheres is trivializable, and the octahedron graph is not trivializable.

Question 1.1. *When is a planar graph trivializable?*

It is known that there exist infinitely many trivializable graphs. Taniyama showed that every *bifocal* as illustrated on the left-hand side in Fig. 1.2 is trivializable [11]. I. Sugiura and S. Suzuki showed that every *3-line web* as illustrated on the right-hand side in Fig. 1.2 is trivializable [8]. N. Tamura showed that every *neo-bifocal* is trivializable and gave a systematic construction of trivializable graphs in terms of an *edge sum* of graphs [9].[1] We refer the reader to [11], [9] and [8] for the precise definitions of the bifocal, the neo-bifocal and the 3-line web, respectively. But trivializable graphs have not been characterized completely yet.

[1] Her method can generate a trivializable graph which is not a minor of a bifocal or neo-bifocal. But the author does not know whether her method generates a trivializable graph which is not a minor of a 3-line web or not.

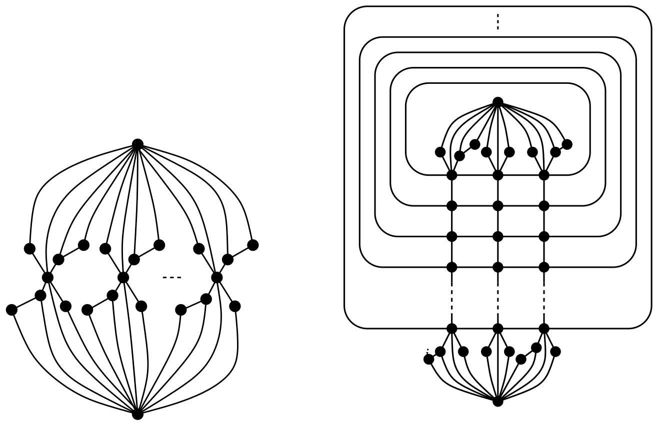

Fig. 1.2. A bifocal and a 3-line web

2. Forbidden graphs for the trivializability

We investigate trivializable graphs from a stand point of graph minor theory. A graph H is called a *minor* of a graph G if H can be obtained from G by a finite sequence of an edge contraction or taking a subgraph. We say that a property $\mathcal{P}$ of a graph is *inherited by minors* if a graph has $\mathcal{P}$ then each proper minor of the graph also has $\mathcal{P}$. Let $\Omega(\mathcal{P})$ be the set of all graphs which do not have $\mathcal{P}$ and whose all proper minors have $\mathcal{P}$. This set is called the *obstruction set* for $\mathcal{P}$ and each of the elements in $\Omega(\mathcal{P})$ is called a *forbidden graph* for $\mathcal{P}$. It is clear that a graph G has $\mathcal{P}$ if and only if G does not have a minor which belongs to $\Omega(\mathcal{P})$. Then, according to N. Robertson-P. Seymour's Graph Minor Theorem [5], the following fact holds.

Theorem 2.1. (Robertson-Seymour [5]) $\Omega(\mathcal{P})$ *is a finite set.* □

In particular for the trivializability, it is known the following.

Proposition 2.2. (Taniyama [11]) *The trivializability is inherited by minors.* □

Therefore by Theorem 2.1, we have that $\Omega(\mathcal{T})$ is a finite set, namely the trivializability of a graph can be determined by finitely many forbidden graphs. Thus we would like to determine all elements in $\Omega(\mathcal{T})$, namely we consider the following problem.

Problem 2.3. *Find all forbidden graphs for the trivializability.*

Sugiura and Suzuki found seven forbidden graphs for the trivializability as follows:

Theorem 2.4. (Sugiura-Suzuki [8]) *The seven graphs $G_1, G_2, \ldots, G_7$ as illustrated in Fig. 2.1 belong to $\Omega(\mathcal{T})$.* □

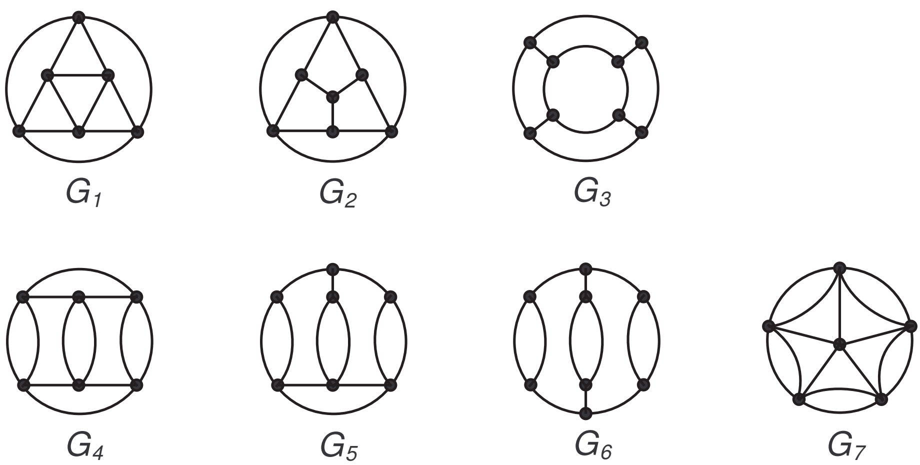

Fig. 2.1. Forbidden graphs $G_1, G_2, \ldots, G_7$ for the trivializability

On the other hand, the author, M. Ozawa, Taniyama and Y. Tsutsumi found nine more forbidden graphs for the trivializability.

Theorem 2.5. (N-Ozawa-Taniyama-Tsutsumi [4]) *The nine graphs $G_8, G_9, \ldots, G_{16}$ as illustrated in Fig. 2.2 belong to $\Omega(\mathcal{T})$.* □

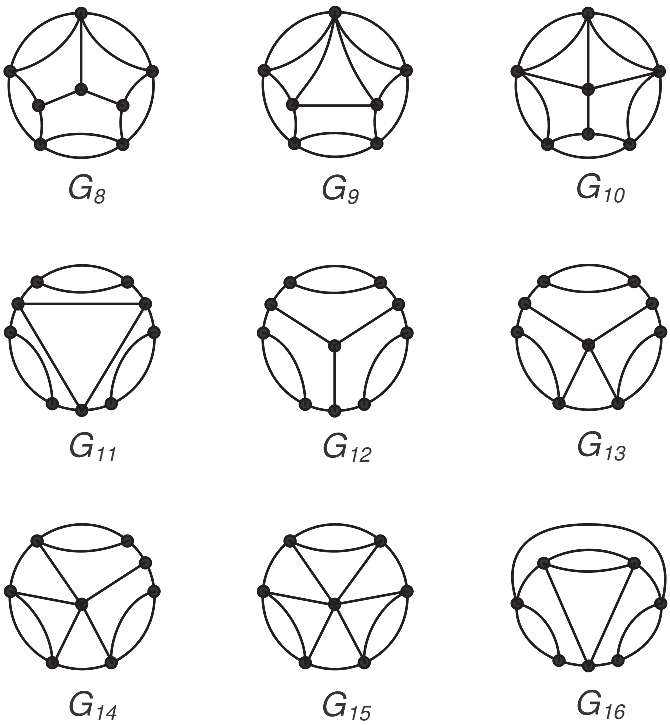

Fig. 2.2. Newly found forbidden graphs $G_8, G_9, \ldots, G_{16}$ for the trivializability

Indeed, each G_i has a knotted projection $\hat{f}_i$ $(i = 8, 9, \ldots, 16)$ as illustrated in Fig. 2.3. Therefore each G_i is not trivializable. Moreover we can see that each of the proper minors of G_i $(i = 8, 9, \ldots, 16)$ is also a minor of a 3-line web. Thus by Proposition 2.2 we have that $G_8, G_9, \ldots, G_{16} \in \Omega(\mathcal{T})$.

It seems that $\Omega(\mathcal{T})$ is not determined by these sixteen forbidden graphs. Actually we have candidates for forbidden graphs for the trivializability. Let H be the graph as illustrated in Fig. 2.4. Sugiura asked in his master thesis [7] whether H is trivializable or not. This question is still open. Since we can see that each of the proper minors of H is trivializable, if H is not trivializable then $H \in \Omega(\mathcal{T})$.

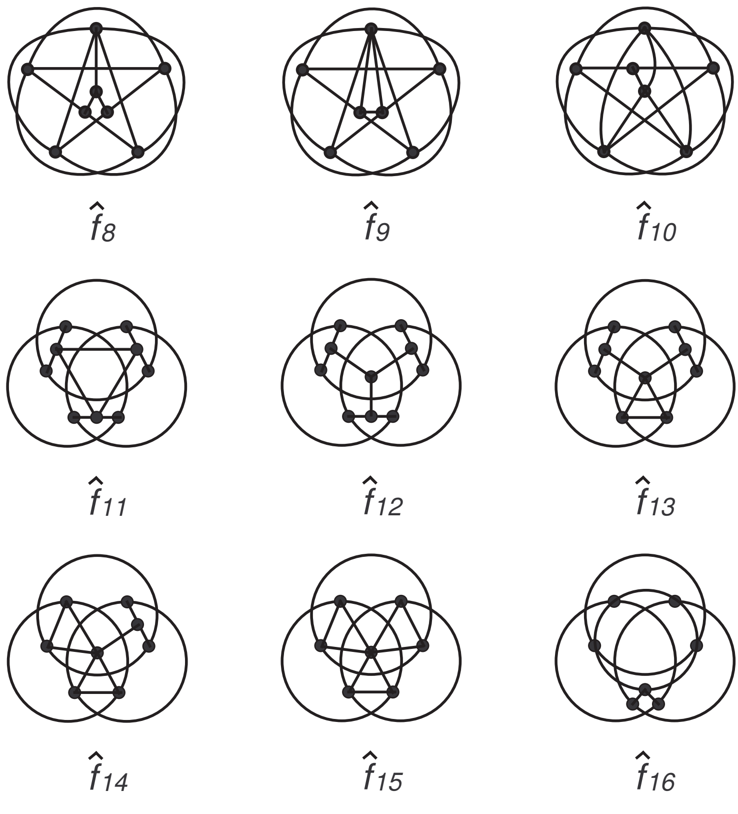

Fig. 2.3. Knotted projections $\hat{f}_i : G_i \to \mathbf{R}^2$ $(i = 8, 9, \ldots, 16)$

Besides, let H_1, H_2 and H_3 be three graphs as illustrated in Fig. 2.4. Then we have that each H_i has a knotted projection $\hat{g}_i$ $(i = 1, 2, 3)$ as illustrated in Fig. 2.5. We note that each of H_i has a minor which is homeomorphic to H. Thus if H is not trivializable then H_1, H_2 and H_3 are not forbidden graphs for the trivializability, and if H is trivializable then there is a possibility that $H_1, H_2, H_3 \in \Omega(\mathcal{T})$.

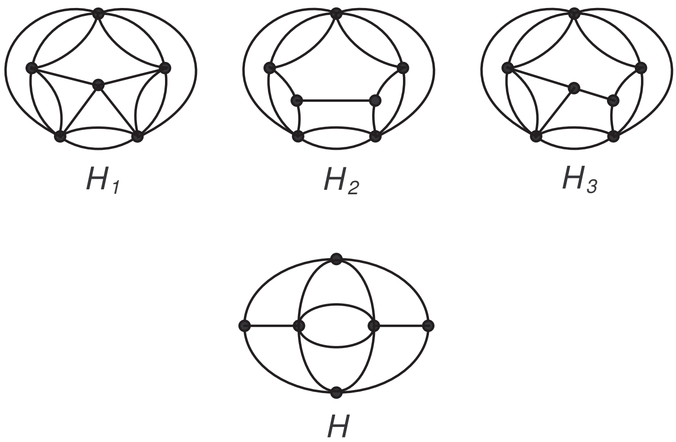

Fig. 2.4.

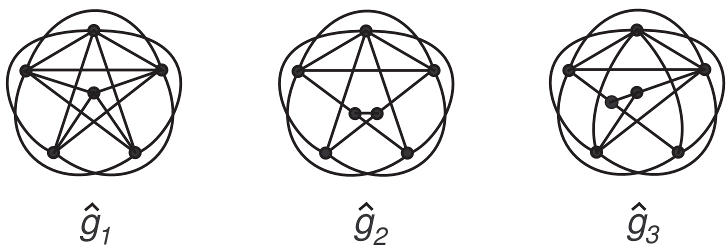

Fig. 2.5. Knotted projections $\hat{g}_i : H_i \to \mathbf{R}^2$ $(i = 1, 2, 3)$

3. Identifiable projections of spatial graphs

A regular projection $\hat{f}$ of a graph is said to be *identifiable* [1] if any two spatial embeddings of the graph obtained from $\hat{f}$ are ambient isotopic. For example, each of the regular projections as illustrated in Fig. 3.1 (1), (2) and (3) is identifiable. We note that a non-planar graph does not have an identifiable projection [1]. Actually this is shown by calculating the *Simon invariant* [10] of spatial subgraph which is homeomorphic to K_5 or $K_{3,3}$.

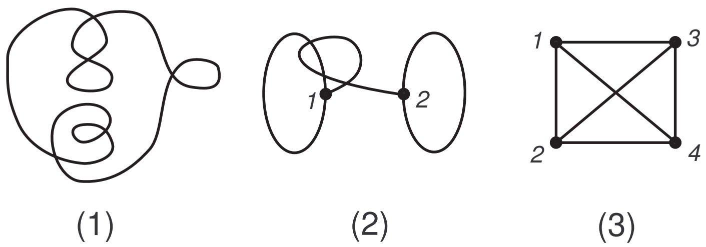

Fig. 3.1. Identifiable projections

Let $\hat{f}$ be an identifiable projection of a trivializable graph G. Then we have that any of the spatial embeddings of G which is obtained from $\hat{f}$ is trivial because there exists a trivial spatial embedding of G which is obtained from $\hat{f}$. But this argument does not work for non-trivializable planar graphs because the projection may be knotted. Thus it is natural to ask the following question.

Question 3.1. *Is any of the spatial embeddings of a non-trivializable planar graph which is obtained from an identifiable projection trivial?*

We give an affirmative answer for Question 3.1, namely we have the following.

Theorem 3.2. (N [3]) *A regular projection of a planar graph is identifiable if and only if any of the spatial embeddings which is obtained from the projection is trivial.*

In the following we give a proof of Theorem 3.2. A spatial embedding f of a graph G is said to be *free* if $\pi_1(\mathbf{R}^3 - f(G))$ is a free group. The following is M. Scharlemann-A. Thompson's famous criterion.

Theorem 3.3. (Scharlemann-Thompson [6]) *For a planar graph G, a spatial embedding f of G is trivial if and only if $\pi_1(\mathbf{R}^3 - f(H))$ is a free group for any subgraph H of G.* □

On the other hand, Ozawa pointed out the following fact.

Lemma 3.4. (N-Ozawa-Taniyama-Tsutsumi [4]) *Let $\hat{f}$ be a regular projection of a graph. Then there exists a free spatial embedding of the graph which is obtained from $\hat{f}$.* □

Proof of Theorem 3.2. By the uniqueness of the trivial spatial embeddings of a planar graph up to ambient isotopy, we have the 'if' part. Next we show the 'only if' part. Let $\hat{f}$ be an identifiable projection of a planar graph G and f the spatial embedding of G obtained from $\hat{f}$. We note that $\hat{f}|_H$ is also identifiable for any subgraph H of G. Then by Lemma 3.4 the spatial embedding g of H obtained from $\hat{f}|_H$ is free. Since $g = f|_H$, we have that $f|_H$ is free for any subgraph H of G. Therefore by Theorem 3.3 we have that f is trivial. This completes the proof. □

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