Classification of spatial complete graphs on four vertices up to C_5-moves

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Dedicated to Professor Kunio Murasugi for his 80th birthday

1. C_k-moves on spatial graphs and finite type invariants

Let $\mathbb{S}^3$ be the unit 3-sphere in $\mathbb{R}^4$ centered at the origin and $\mathbb{S}^2$ the unit 2-sphere in $\mathbb{S}^3$. Let f be an embedding of a finite graph G into $\mathbb{S}^3$. Then f is called a *spatial embedding* of G or simply a *spatial graph*. Two spatial embeddings f and g of G are said to be *ambient isotopic* if there exists an orientation-preserving self homeomorphism Φ on $\mathbb{S}^3$ such that $\Phi \circ f = g$. A graph G is said to be *planar* if there exists an embedding of G into $\mathbb{S}^2$. A spatial embedding f of a planar graph G is said to be *trivial* if there exists an embedding h of G into $\mathbb{S}^2$ such that f and h are ambient isotopic.

A C_1-*move* is a crossing change and a C_k-*move* is a local move on spatial graphs as illustrated in Fig. 1.1 for $k \geq 2$ [4], [2]. Note that a C_2-move is equal to a *delta move* [8], [12], and a C_3-move is equal to a *clasp-pass move* [3]. Two spatial embeddings of a graph are said to be C_k-*equivalent* if they are transformed into each other by C_k-moves and ambient isotopies. By the definition of a C_k-move, it is easy to see that C_k-equivalence implies C_{k-1}-equivalence.

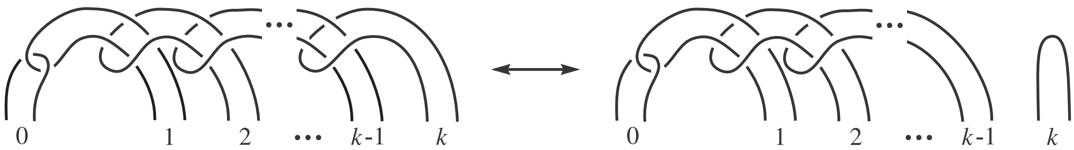

Figure 1.1.

A C_k-move is closely related to *finite type invariants* of knots, links and spatial graphs. For a graph G, we give an orientation to each of the edges of G. A *singular spatial embedding* of G is an immersion of G into $\mathbb{S}^3$ whose multipoints are only transversal double points away from vertices. Let v be an ambient isotopy invariant of spatial graphs taking values in an additive group. We extend v to singular spatial embeddings of G by $v(K_\times) = v(K_+) - v(K_-)$, where $K_\times, K_+$ and K_- are singular spatial embeddings of G which are identical except inside the depicted regions as illustrated in Fig. 1.2. Then v is called a *finite type invariant of order* $\leq n$ if v vanishes on every singular spatial embedding of G with at least $n+1$ double points [18], [1], [14]. If v is of order $\leq n$ but not of order $\leq n-1$, then v is called a *finite type invariant of order* n.

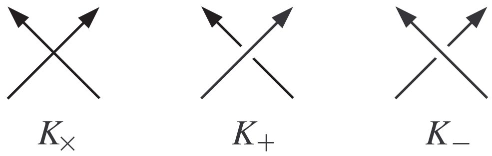

Figure 1.2.

We say that two spatial embeddings f and g of G are FT_n*-equivalent* if $v(f) = v(g)$ for any finite type invariant v of order $\leq n$. In particular for oriented knots, Goussarov and Habiro showed independently the following.

Theorem 1.1. ([2], [4]) *Two oriented knots J and K are C_k-equivalent if and only if they are FT_{k-1}-equivalent.*

The 'only if' part of Theorem 1.1 is also true for oriented links [2], [4] and spatial graphs [17], but the 'if' part does not always hold. For example, the Whitehead link and the trivial 2-component link are FT_2-equivalent but not C_3-equivalent [16]. By finding a basis for the space of finite type invariants for knots, we also have the following.

Theorem 1.2. *Let J and K be two oriented knots. Then we have the following.*

(1) ([12], [8]) *J and K are C_2-equivalent.*
(2) ([3]) *J and K are C_3-equivalent if and only if $a_2(J) = a_2(K)$.*
(3) *J and K are C_4-equivalent if and only if they are C_3-equivalent and $P_0^{(3)}(J;1) = P_0^{(3)}(K;1)$.*
(4) *J and K are C_5-equivalent if and only if they are C_4-equivalent, $a_4(J) = a_4(K)$ and $P_0^{(4)}(J;1) = P_0^{(4)}(K;1)$.*

Here, $a_n(\cdot)$ denotes the nth coefficient of the *Conway polynomial* and $P_m^{(n)}(\cdot;1)$ denotes the nth derivative at 1 of the *HOMFLYPT mth coefficient polynomial* $P_m(\cdot;t)$. For spatial embeddings of a graph which may not be homeomorphic to the circle, a C_k-classification of them has been completed with the comparatively small k. The following table shows the present status of the completion of these classifications.

	C_2	C_3	C_4	C_5	$\cdots$
knots	[8], [12] $C_2 = FT_1$	[4], [2] $C_k = FT_{k-1}$			
2-component links	[12] $C_2 = FT_1$	[16] $C_3 \neq FT_2$	[9] $C_4 = FT_3$	?	
3-component links	[12] $C_2 = FT_1$	[16] $C_3 \neq FT_2$	?		
$k(\geq 4)$-component links	[12] $C_2 = FT_1$	? $C_3 \neq FT_2$	?		
spatial embeddings of planar graphs without disjoint cycles	[15], [11] $C_2 = FT_1$	[16] $C_3 = FT_2$	?		
spatial embeddings of planar graphs with disjoint cycles	[15], [11] $C_2 = FT_1$	?			
spatial embeddings of nonplanar graphs	[15], [11] $C_2 = FT_1$	?			

Let Θ be the *theta curve* and K_4 the *complete graph on four vertices* as illustrated in Fig. 1.3. Note that each of Θ and K_4 is planar and does not contain a pair of disjoint cycles. The following are C_2 and C_3-classifications of spatial embeddings of such graphs.

Theorem 1.3. *Let G be a planar graph which does not contain a pair of disjoint cycles and f and g two spatial embeddings of G. Then we have the following.*

(1) ([15], [11]) *f and g are C_2-equivalent.*

(2) ([16]) *f and g are C_3-equivalent if and only if $a_2(f(\gamma)) = a_2(g(\gamma))$ for any subgraph γ of G which is homeomorphic to the circle.*

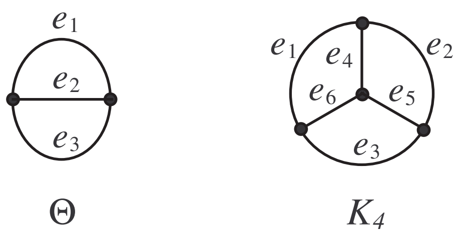

Figure 1.3.

Our purpose in this report is to state classification theorems of spatial theta curves and spatial complete graphs on four vertices under C_4 and C_5-equivalences. For a spatial embedding f of a graph G, a *disk/band surface* S_f of $f(G)$ is a compact and orientable surface in $\mathbb{S}^3$ such that $f(G)$ is a deformation retract of S_f contained in the interior of S_f [7]. In particular, if $G = \Theta$ or K_4, then the disk/band surface of $f(G)$ with zero Seifert linking form is unique with respect to f under ambient isotopy [7], see Fig. 1.4.

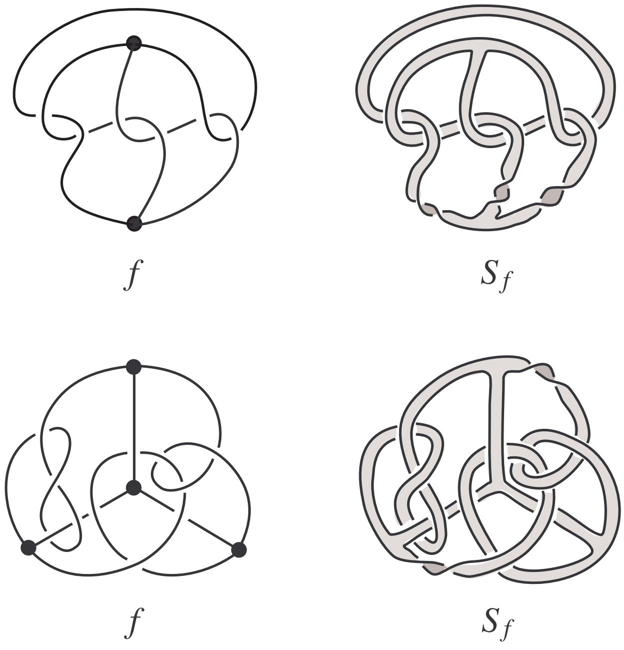

Figure 1.4.

Let $e_1, e_2, \ldots, e_l$ be all edges of G. Let $S_f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l)$ $(\varepsilon_i = 0, \pm 1, \infty)$ be a surface in $\mathbb{S}^3$ obtained from S_f as illustrated in Fig. 1.5. Note that $S_f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l)$ depends only on S_f and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l$. Thus in the case of Θ and K_4, $S_f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l)$ is also the unique surface for f if S_f has zero Seifert linking form. To classify spatial theta curves and spatial complete graphs on four vertices under C_4 and C_5-equivalences, we use some $a_n(\cdot)$ and $P_0^{(n)}(\cdot; 1)$ for knots which appear as the boundary component of the surfaces above. For a knot J, recall that $P(J; t, z)$ does not depend on the orientation of J. Therefore $a_m(J) = P_m(J; 1)$ and $P_m^{(n)}(J; 1)$ also do not depend on the orientation of J.

2. Classification of spatial theta curves and spatial complete graphs on four vertices

First we state complete classifications of spatial theta curves under C_4 and C_5-equivalences. Let f be a spatial theta curve and S_f the disk/band surface of $f(\Theta)$

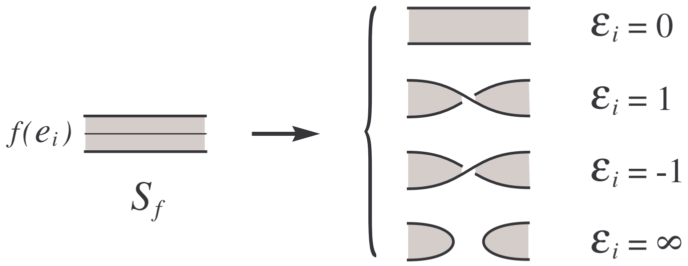

Figure 1.5.

with zero Seifert linking form. We put

$$J_1(f) = f(e_2 \cup e_3),\ J_2(f) = f(e_1 \cup e_3),\ J_3(f) = f(e_1 \cup e_2).$$

Let $J_4(f)$ (resp. $J_5(f)$, $J_6(f)$) be the component of $\partial S_f(0,0,-1)$ (resp. $\partial S_f(-1,0,0)$, $\partial S_f(0,1,0)$) which is not corresponding to $J_3(f)$ (resp. $J_1(f)$, $J_2(f)$). Now we state classification theorems for spatial theta curves under C_4 and C_5-equivalence.

Theorem 2.1. *Two spatial theta curves f and g are C_4-equivalent if and only if the following conditions hold:*

(1) *f and g are C_3-equivalent,*
(2) $P_0^{(3)}(J_i(f);1) = P_0^{(3)}(J_i(g);1)$ $(i = 1,2,3,4)$.

Theorem 2.2. *Two spatial theta curves f and g are C_5-equivalent if and only if the following conditions hold:*

(1) *f and g are C_4-equivalent,*
(2) $a_4(J_i(f)) = a_4(J_i(g))$ $(i = 1,2,3,5)$,
(3) $P_0^{(4)}(J_i(f);1) = P_0^{(4)}(J_i(g);1)$ $(i = 1,2,3,5,6)$.

Example 2.3. There exists a spatial theta curve f such that $J_i(f)$ is trivial for $i = 1,2,3$ but f is not C_4-equivalent to the trivial spatial theta curve h. For example, let f be *Kinoshita's theta curve* as illustrated in Fig. 1.4. It is clear that $J_i(f)$ is trivial for $i = 1,2,3$. But we have $P_0^{(3)}(J_4(f);1) = 48 \neq 0$. Thus f and h are not C_4-equivalent by Theorem 2.1. Note that f and h are C_3-equivalent by Theorem 1.3.

Next we give complete classifications of spatial complete graphs on four vertices under C_4 and C_5-equivalences. Let f be a spatial complete graph on four vertices and S_f the disk/band surface of $f(K_4)$ with zero Seifert linking form. We put

$$J_1(f) = f(e_1 \cup e_2 \cup e_5 \cup e_6),\ J_2(f) = f(e_2 \cup e_3 \cup e_4 \cup e_6),$$
$$J_3(f) = f(e_1 \cup e_3 \cup e_4 \cup e_5),\ J_4(f) = f(e_2 \cup e_4 \cup e_5),$$
$$J_5(f) = f(e_3 \cup e_5 \cup e_6),\ J_6(f) = f(e_1 \cup e_4 \cup e_6),\ J_7(f) = f(e_1 \cup e_2 \cup e_3).$$

Let $J_8(f)$ (resp. $J_9(f)$, $J_{10}(f)$) be the component of $\partial S_f(\infty, 0, 0, 0, -1, 0)$ (resp. $\partial S_f(0, \infty, 0, 0, 0, 1)$, $\partial S_f(0, 0, \infty, -1, 0, 0)$) which is not corresponding to $J_2(f)$ (resp. $J_3(f)$, $J_1(f)$). Let $J_{11}(f)$ (resp. $J_{12}(f)$, $J_{13}(f)$) be the component of $\partial S_f(0, 0, 0, \infty, 1, 0)$ (resp. $\partial S_f(0, 0, 0, 0, \infty, 1)$, $\partial S_f(0, 0, 0, -1, 0, \infty)$) which is not corresponding to $J_7(f)$. Let $J_{14}(f)$ (resp. $J_{15}(f)$) be the component of $\partial S_f(\infty, -1, 0, 0, 0, 0)$ (resp. $\partial S_f(-1, \infty, 0, 0, 0, 0)$) which is not corresponding to $J_5(f)$. Let $J_{16}(f)$ (resp. $J_{17}(f)$) be the component of $\partial S_f(-1, 0, \infty, 0, 0, 0)$ (resp. $\partial S_f(\infty, 0, 1, 0, 0, 0)$) which is not corresponding to $J_4(f)$. Let $J_{18}(f)$ (resp. $J_{19}(f)$) be the component of $\partial S_f(0, \infty, 1, 0, 0, 0)$ (resp. $\partial S_f(0, 1, \infty, 0, 0, 0)$) which is not corresponding to $J_6(f)$. Let $J_{20}(f)$ (resp. $J_{21}(f)$, $J_{22}(f)$) be the component of $\partial S_f(0, 0, 0, 0, -1, 1)$ (resp. $\partial S_f(0, 0, 0, -1, -1, 0)$, $\partial S_f(0, 0, 0, -1, 0, 1)$) which is not corresponding to $J_7(f)$. Let $J_{23}(f)$ (resp. $J_{24}(f)$) be the component of $\partial S_f(1, 0, 0, 0, \infty, 0)$ (resp. $\partial S_f(0, 1, 0, 0, 0, \infty)$) which is not corresponding to $J_2(f)$ (resp. $J_3(f)$). Now we state classification theorems for spatial complete graphs on four vertices under C_4 and C_5-equivalence.

Theorem 2.4. *Two spatial complete graphs on four vertices f and g are C_4-equivalent if and only if the following conditions hold:*

(1) *f and g are C_3-equivalent,*
(2) $P_0^{(3)}(J_i(f); 1) = P_0^{(3)}(J_i(g); 1)$ $(i = 1, 2, \dots, 13)$.

Let Θ_i be the subgraph of Θ which is obtained from Θ by deleting the edge e_i $(i = 1, 2, \dots, 6)$. Note that Θ_i is homeomorphic to Θ. Then, by Theorems 2.1 and 2.4, we have the following.

Corollary 2.5. *Two spatial complete graphs on four vertices f and g are C_4-equivalent if and only if $f|_{\Theta_i}$ and $g|_{\Theta_i}$ are C_4-equivalent* $(i = 1, 2, \dots, 6)$.

Example 2.6. Let f and g be two spatial complete graphs on four vertices as illustrated in Fig. 2.1. Since $J_i(f)$ is trivial for $i = 1, 2, \dots, 7$, by Theorem 1.3 it follows that f is C_3-equivalent to the trivial spatial complete graph on four vertices h. But we can see that $f|_{\Theta_1}$ is the Kinoshita's theta curve. Thus by Example 2.3 and Corollary 2.5, f and h are not C_4-equivalent. On the other hand, we can see that $g|_{\Theta_i}$ is trivial for $i = 1, 2, \dots, 6$. Thus by Corollary 2.5, it follows that g and h are C_5-equivalent. Note that g is not trivial under ambient isotopy.

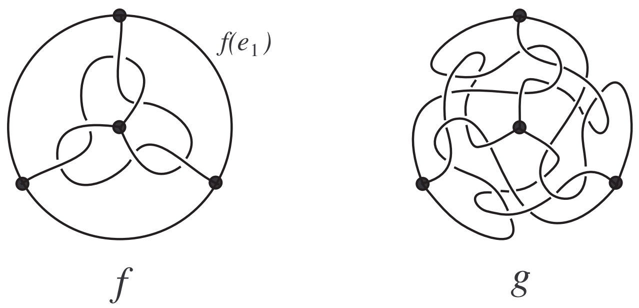

Figure 2.1.

Theorem 2.7. *Two spatial complete graphs on four vertices f and g are C_5-equivalent if and only if the following conditions hold:*

(1) *f and g are C_4-equivalent,*
(2) $a_4(J_i(f)) = a_4(J_i(g))$ $(i = 1, 2, \ldots, 16)$,
(3) $P_0^{(4)}(J_i(f); 1) = P_0^{(4)}(J_i(g); 1)$ $(i = 1, 2, \ldots, 24)$.

Example 2.8. There exists a spatial complete graph on four vertices f such that $f|_{\Theta_i}$ is C_5-equivalent to the trivial spatial embedding of Θ_i for $i = 1, 2, \ldots, 6$ but f is not C_5-equivalent to the trivial spatial complete graph on four vertices h. For example, let f be the spatial complete graph on four vertices as illustrated in Fig. 2.2. We can see that $f|_{\Theta_i}$ is trivial for $i = 1, 2, \ldots, 5$. Though $f|_{\Theta_6}$ is not trivial, by checking the conditions in Theorem 2.2, we can see that $f|_{\Theta_6}$ is C_5-equivalent to the trivial spatial embedding of Θ_6. But we have $P_0^{(4)}(J_{20}(f); 1) = 384 \neq 0$. Thus f and h are not C_5-equivalent by Theorem 2.7.

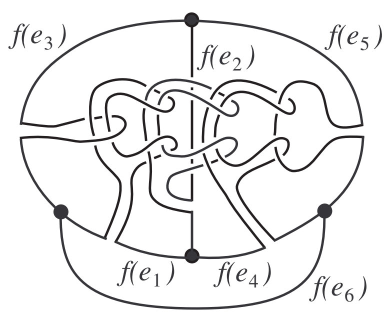

Figure 2.2.

Question 2.9. *Does there exist a spatial complete graph on four vertices f such that $f|_{\Theta_i}$ is trivial for $i = 1, 2, \ldots, 6$ but f is not C_5-equivalent to the trivial spatial complete graph on four vertices?*

Let f be a spatial theta curve (resp. spatial complete graph on four vertices) and S_f the disk/band surface of $f(\Theta)$ (resp. $f(K_4)$) with zero Seifert linking form. Then it is known that a finite type invariant of order $\leq n$ of $\partial S_f(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ (resp. $\partial S_f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_6)$) is also a finite type invariant of order $\leq n$ of f [13], [19]. On the other hand, $a_n(\cdot)$ is a finite type invariant of order $\leq n$ [1], and $P_0^{(n)}(\cdot; 1)$ is a finite type invariant of order $\leq n$ [6]. Therefore, by Theorems 1.3, 2.1, 2.2, 2.4 and 2.7, we have the following.

Corollary 2.10. *Let $G = \Theta$ or K_4. For $k \leq 5$, two spatial embeddings f and g of G are C_k-equivalent if and only if they are FT_{k-1}-equivalent.*

Remark 2.11.

(1) Proofs of Theorems 2.1, 2.2, 2.4 and 2.7 are done by showing a slightly modified version of Meilhan and the second author's C_4 and C_5-classifications of string links [10].

(2) Let H be the *handcuff graph*, which is constructed by connecting two loops by a single edge. Then, there exists a spatial handcuff graph f such that f is FT_{k-1}-equivalent to the trivial spatial handcuff graph h but not C_k-equivalent to h for $k = 3, 4, 5$. Classification of spatial handcuff graphs under C_k-equivalence for $k = 3, 4, 5$ are due to be mentioned in [5].

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