On boundary spatial embeddings of a graph

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Abstract

A spatial embedding of a graph is called a *∂-spatial embedding* if all knots in the embedding bound Seifert surfaces simultaneously such that the interiors of the surfaces are mutually disjoint and disjoint from the image of the embedding. This is a generalization of the boundary link. In this paper, we show the following: (1) We give a complete characterization of a graph which has a ∂-spatial embedding, (2) We classify ∂-spatial embeddings completely up to *self pass-moves* and ambient isotopies. In particular, any ∂-spatial embedding of a graph is trivial up to edge-homotopy. This result is a generalization of the fact that any boundary link is trivial up to link-homotopy.

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1. Introduction

Let G be a finite graph which does not have free vertices. We denote the set of all vertices and the set of all edges by $V(G)$ and $E(G)$, respectively. We consider G as a topological space in the usual way. An embedding $f : G \to S^3$ is called a *spatial embedding of* G or simply a *spatial graph*. A graph G is said to be *planar* if there exists an embedding of G into S^2. A spatial embedding of a planar graph is said to be *trivial* if it is ambient isotopic to an embedding into $S^2 \subset S^3$.

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A *path* of G is a subgraph of G which is homeomorphic to the closed interval and a *cycle* of G is a subgraph of G which is homeomorphic to S^1. We denote the set of all cycles of G by $\Gamma(G)$. For a spatial embedding f of G and a cycle $\gamma \in \Gamma(G)$, we can regard $f(\gamma)$ as a knot in the spatial embedding. We set $\Gamma(G) = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$. We call a spatial embedding f of G a *∂-spatial embedding* if there exist compact, connected and orientable surfaces $S_1, S_2, \ldots, S_n$ in S^3 such that
(1) $f(G) \cap S_i = f(G) \cap \partial S_i = f(\gamma_i)$ $(i = 1, 2, \ldots, n)$,
(2) $\text{int}S_i \cap \text{int}S_j = \emptyset$ for $i \neq j$.
We note that if G is homeomorphic to the disjoint union of 1-spheres, then a ∂-spatial embedding is a *boundary link* [15].

Example 1.1. Let f be a spatial theta curve as illustrated in Figure 1.1. Then it is easy to see that there exist Seifert surfaces S_1 and S_2 for $f(e_1) \cup f(e_3)$ and $f(e_2) \cup f(e_3)$ respectively such that $S_1 \cap S_2 = f(e_3)$. We note that $S_1 \cup S_2$ is a connected, compact and orientable surface. We define $S_3 = (S_1 \cup S_2)^+$, where $S^{\pm}$ denotes a parallel copy of a connected, compact and oriented surface S with boundary in S^3 obtained by pushing S slightly in the positive (resp. negative) normal direction of S relative to ∂S, namely $S \cap S^{\pm} = \partial S = \partial S^{\pm}$ and $\text{int}S \cap \text{int}S^{\pm} = \emptyset$. Then we have that S_3 is also a Seifert surface for $f(e_1) \cup f(e_2)$ and the interiors of S_1, S_2 and S_3 are mutually disjoint. Therefore we have that f is a ∂-spatial embedding.

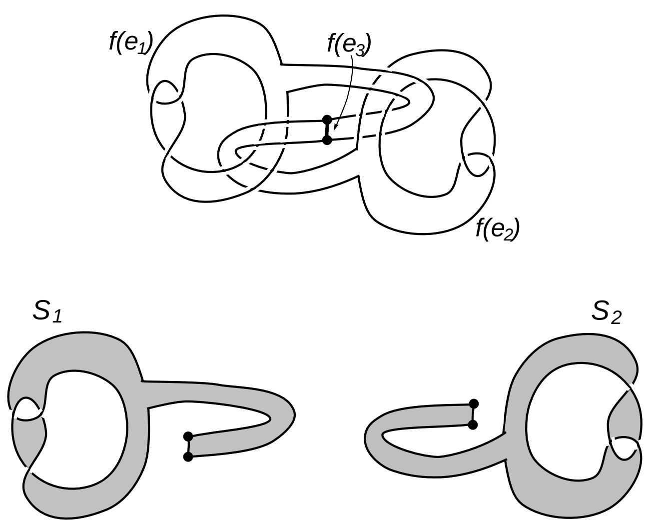

Fig. 1.1.

Every graph does not always have a ∂-spatial embedding. We give a complete characterization of a graph which has a ∂-spatial embedding as follows. A vertex $v \in V(G)$ is called a *cut-vertex of* G if there exist subgraphs H_1 and H_2 of G such that $E(H_i) \neq \emptyset$ $(i = 1, 2)$, $G = H_1 \cup H_2$ and $H_1 \cap H_2 = v$. A graph G is called a *block* if it is connected and does not contain a cut-vertex. A subgraph H of G is called a *block of* G if H is a block and there does not exist a subgraph H' of G such that H' is a block and H is a proper subgraph of H'. For any graph G, it is easy to see that there exist blocks $B_1, B_2, \ldots, B_n$ of G such that $G = B_1 \cup B_2 \cup \cdots \cup B_n$, and the decomposition is essentially unique. We call this decomposition the *block decomposition* of G.

For an edge $e \in E(G)$ that is not a loop, the *edge-contraction* G/e is the graph obtained from $G - \mathrm{int}e$ by identifying the ends of e. A graph H is called a *minor* of G, denoted by $H < G$, if there exists a subgraph G' of G and $e_1, e_2, \ldots, e_m \in E(G')$ such that $H = (\cdots((G'/e_1)/e_2)/\cdots)/e_m$.

In section 2, we prove the following characterization.

Theorem 1.2. *Let $G = B_1 \cup B_2 \cup \cdots \cup B_n$ be a graph and its block decomposition. Then the following are equivalent.*
(1) *There exists a ∂-spatial embedding of G.*
(2) *Each B_i is an edge or a graph which is homeomorphic to one of the graphs $G_1, G_2, \ldots, G_5$ as illustrated in Figure 1.2.*
(3) *Each B_i $(i = 1, 2, \ldots, n)$ does not have a minor which is homeomorphic to one of the graphs G'_1, G'_2 and G'_3 as illustrated in Figure 1.3.*

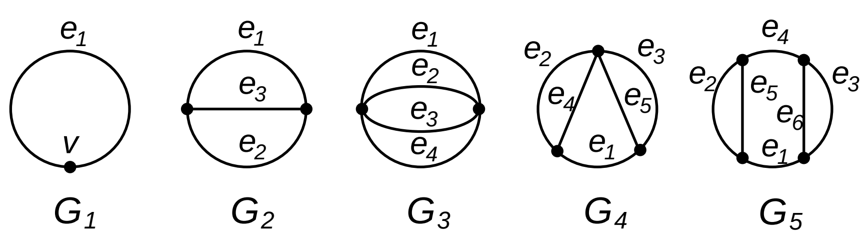

Fig. 1.2.

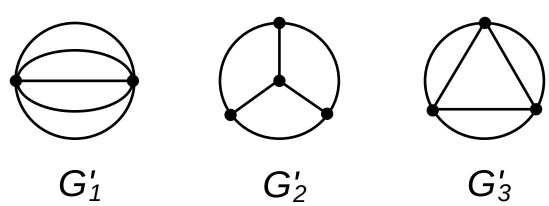

Fig. 1.3.

It is well known that a graph is not planar if and only if it contains a subgraph which is homeomorphic to K_5 or $K_{3,3}$ as illustrated in Figure 1.4 [4]. Since each of K_5 and $K_{3,3}$ has a subgraph which is homeomorphic to G'_2, we have the following.

Corollary 1.3. *Any non-planar graph does not have a ∂-spatial embedding.* $\square$

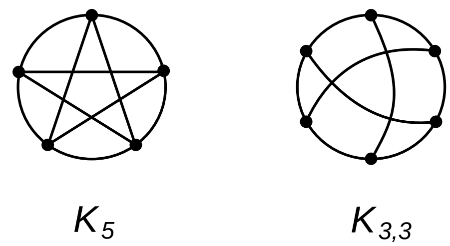

Fig. 1.4.

Let G be an oriented graph, namely an orientation is given to each edge of G. For a spatial embedding f of G, we give the orientation to each spatial edge induced by G. A *pass-move* [3] and a *sharp-move* [7] on a spatial graph are local moves which are illustrated in Figures 1.5 and 1.6, respectively. We also refer the reader to [9] for a related work.

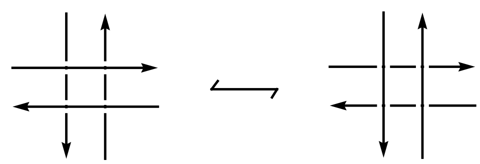

Fig. 1.5.

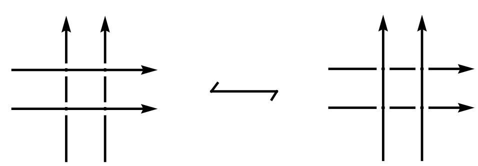

Fig. 1.6.

In this paper we consider a specific pass-move (resp. sharp-move) on a spatial graph. We call a pass-move (resp. sharp-move) on a spatial graph is a *self pass-move* (resp. *self sharp-move*) [12] if all four strings in the move belong to the same spatial edge. We say that two spatial embeddings f and g of G are *self pass-equivalent* (resp. *self sharp-equivalent*) if they are transformed into each other by self pass-moves (resp. self sharp-moves) and ambient isotopies. It is easy to see that these equivalences do not depend on the choice of orientations of edges of G. In particular for oriented links, the following results are known.

Theorem 1.4. (1) (H. Murakami [7]) *Any two oriented knots are self sharp-equivalent.*
(2) (L. H. Kauffman [3]) *Two oriented knots J and K are self pass-equivalent if and only if* $\mathrm{Arf}(J) = \mathrm{Arf}(K)$*, where* $\mathrm{Arf}(\cdot)$ *denotes the Arf invariant* [10].
(3) (T. Shibuya [11]) *Any two boundary links are self sharp-equivalent.*
(4) (L. Cervantes and R. A. Fenn [1]) *Two boundary links* $L = J_1 \cup J_2 \cup \cdots \cup J_n$ *and* $M = K_1 \cup K_2 \cup \cdots \cup K_n$ *are self pass-equivalent if and only if* $\mathrm{Arf}(J_i) = \mathrm{Arf}(K_i)$ $(i = 1, 2, \ldots, n)$. □

We note that $\mathrm{Arf}(K)$ coincides with the modulo two reduction of the second coefficient of the *Conway polynomial* of a knot K [3]. We extend Theorem 1.4 to ∂-spatial embeddings of a graph G as follows.

Theorem 1.5. (1) *Any two ∂-spatial embeddings of a graph are self sharp-equivalent.*
(2) *Two ∂-spatial embeddings f and g of G are self pass-equivalent if and only if* $\mathrm{Arf}(f(\gamma)) = \mathrm{Arf}(g(\gamma))$ *for any* $\gamma \in \Gamma(G)$.

Two spatial embeddings of a graph G are said to be *edge-homotopic* [16] if they are transformed into each other by *self crossing changes* and ambient isotopies, where a self crossing change is a crossing change on the same spatial edge. This is a generalization of *link-homotopy* on oriented links in the sense of J. Milnor [6]. Since a self sharp-move is realized by self crossing changes, we have the following by Theorem 1.5 (1) and Corollary 1.3 as a generalization of the fact that any boundary link is trivial up to link-homotopy [1, 2].

Corollary 1.6. *Any ∂-spatial embedding of a graph is trivial up to edge-homotopy.* □

We prove Theorem 1.5 in section 3. We remark here that all oriented links were classified up to self pass-equivalence by Shibuya and A. Yasuhara in terms of the Arf invariant of proper sublinks and link-homotopy [13].

2. Characterization of graphs which have a ∂-spatial embedding

To prove Theorem 1.2, we recall the following.

Proposition 2.1. ([14]) *Let G be a block with $\beta_1(G) \geq 2$, where $\beta_1(G)$ denotes the first Betti number of G. For cycles $\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma(G)$ and a spatial embedding f of G, if there exist compact, connected and orientable surfaces $S_1, S_2, \ldots, S_k$ in S^3 such that*
(1) $f(G) \cap S_i = f(G) \cap \partial S_i = f(\gamma_i)$ $(i = 1, 2, \ldots, k)$ *and ,*
(2) $\mathrm{int}S_i \cap \mathrm{int}S_j = \emptyset$ *for $i \neq j$,*
then $k \leq 3\beta_1(G) - 3$. □

Actually we can show Proposition 2.1 by counting the number of non-parallel essential simple closed curves in the boundary of the spatial graph-exterior. As a corollary of Proposition 2.1, we have the following.

Corollary 2.2. *Let G be a block with $\beta_1(G) \geq 2$ and $|\Gamma(G)| = n$. If $n > 3\beta_1(G) - 3$, then G does not have a ∂-spatial embedding.* □

For graphs $G_1, G_2, \ldots, G_5$ as illustrated in Figure 1.2, we set

$$\begin{aligned}
\Gamma(G_1) &= \{\gamma_1\}, \\
&\quad \gamma_1 = e_1, \\
\Gamma(G_2) &= \{\gamma_1, \gamma_2, \gamma_3\}, \\
&\quad \gamma_1 = e_1 \cup e_3,\ \gamma_2 = e_2 \cup e_3,\ \gamma_3 = e_1 \cup e_2, \\
\Gamma(G_3) &= \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}, \\
&\quad \gamma_1 = e_1 \cup e_4,\ \gamma_2 = e_2 \cup e_4,\ \gamma_3 = e_3 \cup e_4, \\
&\quad \gamma_4 = e_1 \cup e_2,\ \gamma_5 = e_1 \cup e_3,\ \gamma_6 = e_2 \cup e_3, \\
\Gamma(G_4) &= \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}, \\
&\quad \gamma_1 = e_1 \cup e_4 \cup e_5,\ \gamma_2 = e_2 \cup e_4,\ \gamma_3 = e_3 \cup e_5, \\
&\quad \gamma_4 = e_1 \cup e_2 \cup e_5,\ \gamma_5 = e_1 \cup e_4 \cup e_3,\ \gamma_6 = e_1 \cup e_2 \cup e_3, \\
\Gamma(G_5) &= \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}, \\
&\quad \gamma_1 = e_1 \cup e_4 \cup e_5 \cup e_6,\ \gamma_2 = e_2 \cup e_5,\ \gamma_3 = e_3 \cup e_6, \\
&\quad \gamma_4 = e_1 \cup e_2 \cup e_4 \cup e_6,\ \gamma_5 = e_1 \cup e_3 \cup e_4 \cup e_5, \\
&\quad \gamma_6 = e_1 \cup e_2 \cup e_3 \cup e_4.
\end{aligned}$$

Lemma 2.3. *Let G be a block. Then the following are equivalent.*
(1) *There exists a ∂-spatial embedding of G.*
(2) *G is homeomorphic to one of the graphs $G_1, G_2, \ldots, G_5$ as illustrated in Figure 1.2.*
(3) *G does not have a minor which is homeomorphic to one of the graphs G'_1, G'_2 and G'_3 as illustrated in Figure 1.3.*

Proof. We first show (3) $\Rightarrow$ (2). For a graph $H = H_1 \cup H_2$, we say that *H is obtained from H_1 by a path-addition* if H_2 is a path of H and $H_1 \cap H_2$ is the end points of H_2. It is well known that any block is homeomorphic to a graph which can be obtained from G_1 by path-additions. Then it can be easily seen that G_3, G_4, G_5 and G'_2 are all blocks which can be obtained from G_1 by two path-additions, see Figure 2.1. Then we can check that any of the graphs which can be obtained from G_3, G_4 and G_5 by a path-addition has a minor which is homeomorphic to G'_1, G'_2 or G'_3. Thus we have the result.

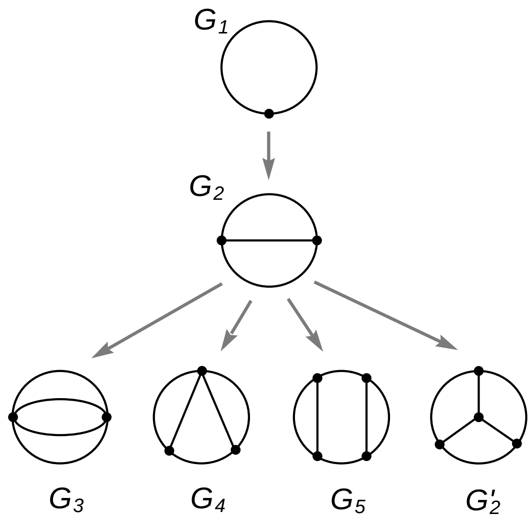

Fig. 2.1.

Next we show (2) $\Rightarrow$ (1). It is sufficient to show that each of G_1, G_2, ..., G_5 has a ∂-spatial embedding. Let B_1 and B_2 be 3-balls such that $S^3 = B_1 \cup B_2$ and $\partial B_1 = \partial B_2 = S^2$. We regard Figure 1.2 as a trivial spatial embedding $h_i : G_i \to S^2 = \partial B_1 = \partial B_2 \subset S^3$ $(i = 1, 2, \ldots, 5)$. It is clear that h_1 is a ∂-spatial embedding, namely there exists a 2-disk D_1 in S_2 such that $\partial D_1 = h_1(\gamma_1)$. Next we consider h_2. There exist 2-disks D_1 and D_2 in S^2 such that $\partial D_i = h_2(\gamma_i)$ $(i = 1, 2)$. Besides we can obtain a 2-disk D_3 which is properly embedded in B_1 such that $\partial D_3 = h_2(\gamma_3)$. Since D_1, D_2 and D_3 have mutually disjoint interiors, we have that h_2 is a ∂-spatial embedding. Next we consider h_3. There exist 2-disks D_3, D_4 and D_6 in S^2 such that $\partial D_i = h_3(\gamma_i)$ $(i = 3, 4, 6)$. Besides we can obtain a 2-disk D_5 which is properly embedded in B_1 such that $\partial D_5 = h_3(\gamma_5)$ and a 2-disk D_2 which is properly embedded in B_2 such that $\partial D_2 = h_3(\gamma_2)$. Then $S^2 \cup D_5 - \mathrm{int}(D_4 \cup D_6)$ is a 2-sphere in B_1 which bounds a 3-ball B_3 and we can obtain a 2-disk D_1 which is properly embedded in B_3 such that $\partial D_1 = h_3(\gamma_1)$. Since $D_1, D_2, \ldots, D_6$ have mutually disjoint interiors, we have that h_3 is a ∂-spatial embedding. We have that h_4 and h_5 are ∂-spatial embeddings in the same way as the case of h_3. Thus we have the result.

Finally we show (1) $\Rightarrow$ (3). Assume that G has a ∂-spatial embedding f. For any subgraph H of G, it is easy to see that $f|_H$ is a ∂-spatial embedding of H. Let e be an edge of G that is not a loop. Then the contraction of e induces a bijection from $\Gamma(G)$ to $\Gamma(G/e)$ and we can see that $f(G)/f(e)$ represents a ∂-spatial embedding of G/e naturally. Therefore we have that each minor of G has a ∂-spatial embedding. But we can see that each of G'_1, G'_2 and G'_3 is a block and does not have a ∂-spatial embedding by Corollary 2.2. Thus G cannot have a minor which is homeomorphic to G'_1, G'_2 or G'_3. This completes the proof. □

Proof of Theorem 1.2. By considering the block decomposition of any graph, we have the result immediately by Lemma 2.3. □

3. Classification of ∂-spatial embeddings of a graph up to self pass-equivalence

It is known that a pass-move is realized by sharp-moves and ambient isotopies as illustrated in Figure 3.1 [8]. Thus we have the following.

Lemma 3.1. *Self-pass equivalence implies self sharp-equivalence.* □

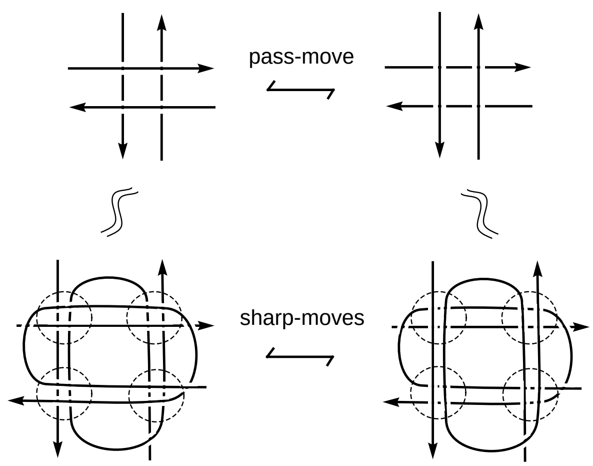

Fig. 3.1.

A Γ-*move* [3] is a local move on a spatial graph as illustrated in Figure 3.2. We call a Γ-move a *self* Γ-*move* if all three strings in the move belong to the same spatial edge. It is known that a Γ-move is realized by a pass-move [3], see Figure 3.3. Thus we have the following.

Lemma 3.2. *A self* Γ*-move is realized up to self pass-equivalence.* □

Lemma 3.3. *If two spatial embeddings f and g of G are self pass-equivalent, then* $\mathrm{Arf}(f(\gamma)) = \mathrm{Arf}(g(\gamma))$ *for any* $\gamma \in \Gamma(G)$.

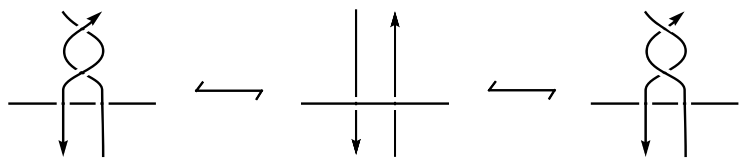

Fig. 3.2.

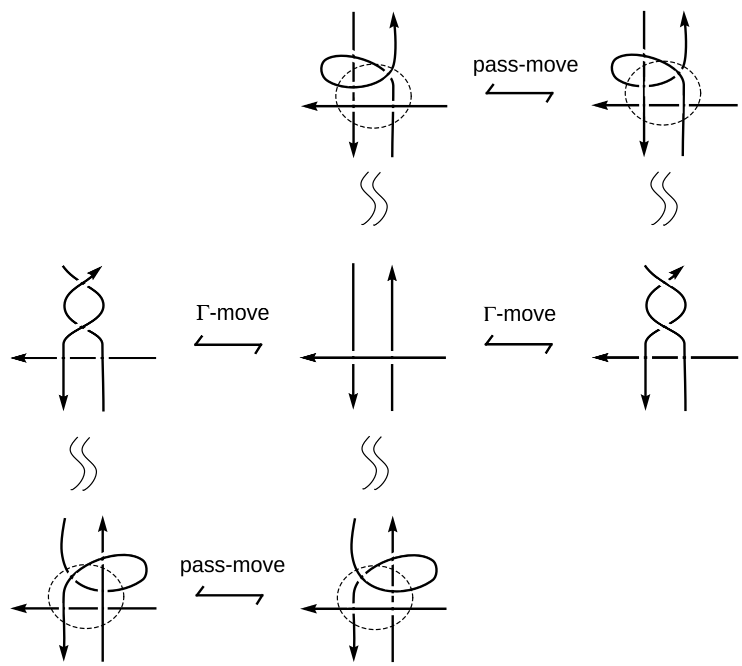

Fig. 3.3.

Proof. If two spatial embeddings f and g of G are self pass-equivalent, it is clear that $f(\gamma)$ and $g(\gamma)$ are self pass-equivalent for any $\gamma \in \Gamma(G)$. Thus by Theorem 1.4 (2) we have that $\mathrm{Arf}(f(\gamma)) = \mathrm{Arf}(g(\gamma))$. □

Proof of Theorem 1.5. We first prove (2). By Lemma 3.3, we have the 'only if' part. So we show the 'if' part. Let f and g be ∂-spatial embeddings of G such that $\mathrm{Arf}(f(\gamma)) = \mathrm{Arf}(g(\gamma))$ for any $\gamma \in \Gamma(G)$. In the following we show that f can be transformed into a canonical spatial embedding ψ_f up to self pass-equivalence.

Let

$$G = \bigcup_{l=0}^{5} \bigcup_{i_l=1}^{n_l} B_{i_l}^{(l)}$$

be the block decomposition of G such that $B_{i_0}^{(0)}$ is an edge ($i_0 = 1, 2, \ldots, n_0$) and $B_{i_l}^{(l)}$ is homeomorphic to G_l ($i_l = 1, 2, \ldots, n_l$ and $l = 1, 2, \ldots, 5$). We fix a homeomorphism $\varphi_{i_l}^{(l)} : G_l \to B_{i_l}^{(l)}$ and put

$$\Gamma(B_{i_l}^{(l)}) = \left\{ \gamma_{i_l,j}^{(l)} = \varphi_{i_l}^{(l)}(\gamma_j) \;\mid\; \gamma_j \in \Gamma(G_l) \right\}$$

($i_l = 1, 2, \ldots, n_l$ and $l = 1, 2, \ldots, 5$). Let $T_1 = v$, $T_2 = e_3$, $T_3 = e_4$, $T_4 = e_4 \cup e_5$ and $T_5 = e_4 \cup e_5 \cup e_6$ be spanning trees of $G_1, G_2, \ldots, G_5$, respectively. Namely $T_{i_l}^{(l)} = \varphi_{i_l}^{(l)}(T_l)$ is a spanning tree of $B_{i_l}^{(l)}$.

Since f is a ∂-spatial embedding of G, there exist connected, compact and orientable surfaces $S_{i_l,j}^{(l)}$ ($l = 1, 2, \ldots, 5,\ i_l = 1, 2, \ldots, n_l,\ j = 1, 2, \ldots$) such that the interiors of them are mutually disjoint and

$$f(G) \cap S_{i_l,j}^{(l)} = f(G) \cap \partial S_{i_l,j}^{(l)} = f(\gamma_{i_l,j}^{(l)}).$$

Let us consider

$$P = f(G) \cup \bigcup_{i_1=1}^{n_1} S_{i_1,1}^{(1)} \cup \bigcup_{i_2=1}^{n_2} \left(\bigcup_{j=1}^{2} S_{i_2,j}^{(2)} \right) \cup \bigcup_{i_3=1}^{n_3} \left(\bigcup_{j=1}^{3} S_{i_3,j}^{(3)} \right) \cup \bigcup_{i_4=1}^{n_4} \left(\bigcup_{j=1}^{3} S_{i_4,j}^{(4)} \right) \cup \bigcup_{i_5=1}^{n_5} \left(\bigcup_{j=1}^{3} S_{i_5,j}^{(5)} \right).$$

Let $N_{i_1}^{(1)}, N_{i_2}^{(2)}, N_{i_3}^{(3)}, N_{i_4}^{(4)}$ and $N_{i_5}^{(5)}$ be regular neighbourhoods of $f(T_{i_1}^{(1)})$, $f(T_{i_2}^{(2)})$, $f(T_{i_3}^{(3)})$, $f(T_{i_4}^{(4)})$ and $f(T_{i_5}^{(5)})$ in $S_{i_1,1}^{(1)}$, $S_{i_2,1}^{(2)} \cup S_{i_2,2}^{(2)}$, $S_{i_3,1}^{(3)} \cup S_{i_3,2}^{(3)} \cup S_{i_3,3}^{(3)}$, $S_{i_4,1}^{(4)} \cup S_{i_4,2}^{(4)} \cup S_{i_4,3}^{(4)}$ and $S_{i_5,1}^{(5)} \cup S_{i_5,2}^{(5)} \cup S_{i_5,3}^{(5)}$, respectively such that $N_{i_l}^{(l)}$ contains all cut-vertices between $f(B_{i_l}^{(l)})$ and the other blocks of $f(G)$ ($i_l = 1, 2, \ldots, n_l$, $l = 1, 2, \ldots, 5$) as illustrated in Figure 3.4. Then we can regard

$$\bigcup_{l=1}^{5} \bigcup_{i_l=1}^{n_l} \left(f(T_{i_l}^{(l)}) \cup \partial N_{i_l}^{(l)} \right)$$

as a trivial spatial embedding h of G and

$$F = \mathrm{cl}\left(P - \bigcup_{l=1}^{5}\bigcup_{i_l=1}^{n_l} N_{i_l}^{(l)}\right)$$

is the disjoint union of spanning surfaces of a boundary link $L = \partial F$. Therefore we may assume that there exist mutually disjoint $n_1 + 2n_2 + 3n_3 + 3n_4 + 3n_5$ 2-disks $b_{i_l,j}^{(l)}$ embedded in S^3 such that $b_{i_l,j}^{(l)} \cap F = \partial b_{i_l,j}^{(l)} \cap \partial F$ is an arc, $b_{i_l,j}^{(l)} \cap h(G) = \partial b_{i_l,j}^{(l)} \cap \mathrm{int} h(\varphi_{i_l}^{(l)}(e_j))$ is also an arc ($i_l = 1, 2, \ldots, n_l$, $l = 1, 2, \ldots, 5$ and $j = 1, 2, \ldots$) and

$$f(G) = h(G) \cup \bigcup \partial b_{i_l,j}^{(l)} \cup L - \bigcup \mathrm{int}\left(h(\varphi_{i_l}^{(l)}(e_j)) \cap b_{i_l,j}^{(l)}\right) - \mathrm{int}\left(\partial F \cap b_{i_l,j}^{(l)}\right),$$

see Figure 3.5. We call this a *band sum of a boundary link L and $h(G)$*.

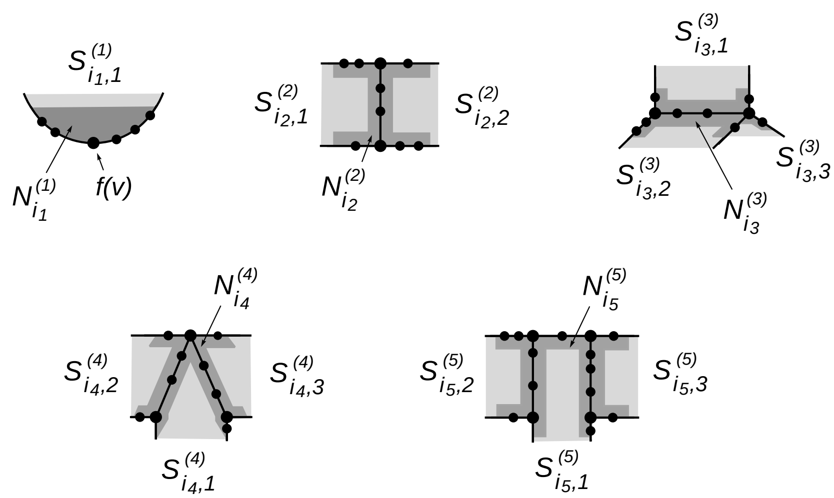

Fig. 3.4.

By Theorem 1.4 (4), L can be transformed into a completely split link L' up to self pass-equivalence such that each of the components of L is a trivial knot or a trefoil knot. Thus we have that f can be transformed into a band sum of L' and $h(G)$ up to self pass-equivalence, see Figure 3.6. Then by using self Γ-moves, namely up to self pass-equivalence by Lemma 3.2, we can shrink each band with the component of L' one by one, see Figure 3.6. By shrinking all bands in such a way, we obtain a spatial embedding ψ_f which is a trivial spatial embedding with some local trefoil knots. We note that a local trefoil knot attached to $\psi_f(\varphi_{i_l}^{(l)}(e_j))$ is unique up to ambient isotopy. We have that g also can be transformed into a canonical spatial embedding ψ_g up to self pass-equivalence in the same way. Since a trivial spatial embedding of a planar graph is unique up to ambient isotopy [5],

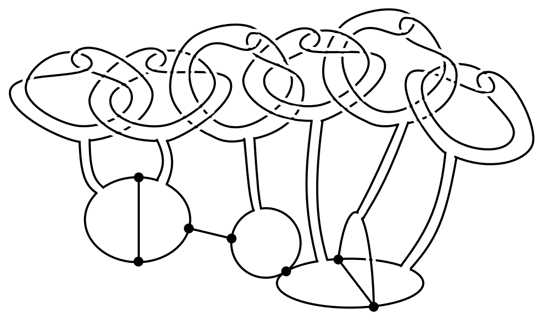

Fig. 3.5.

by the assumption we have that $\psi_f = \psi_g$. Therefore we have that f and g are self pass-equivalent.

Next we prove (1). By Lemma 3.1, we have that any ∂-spatial embedding f of a graph can be transformed into ψ_f by self sharp-equivalence in the same way as the proof of (2). We note that the self sharp-move is an unknotting operation (Theorem 1.4 (1)). Thus we can undone each of the local knots by self sharp-moves. So we have that f is trivial up to self sharp-equivalence. $\square$

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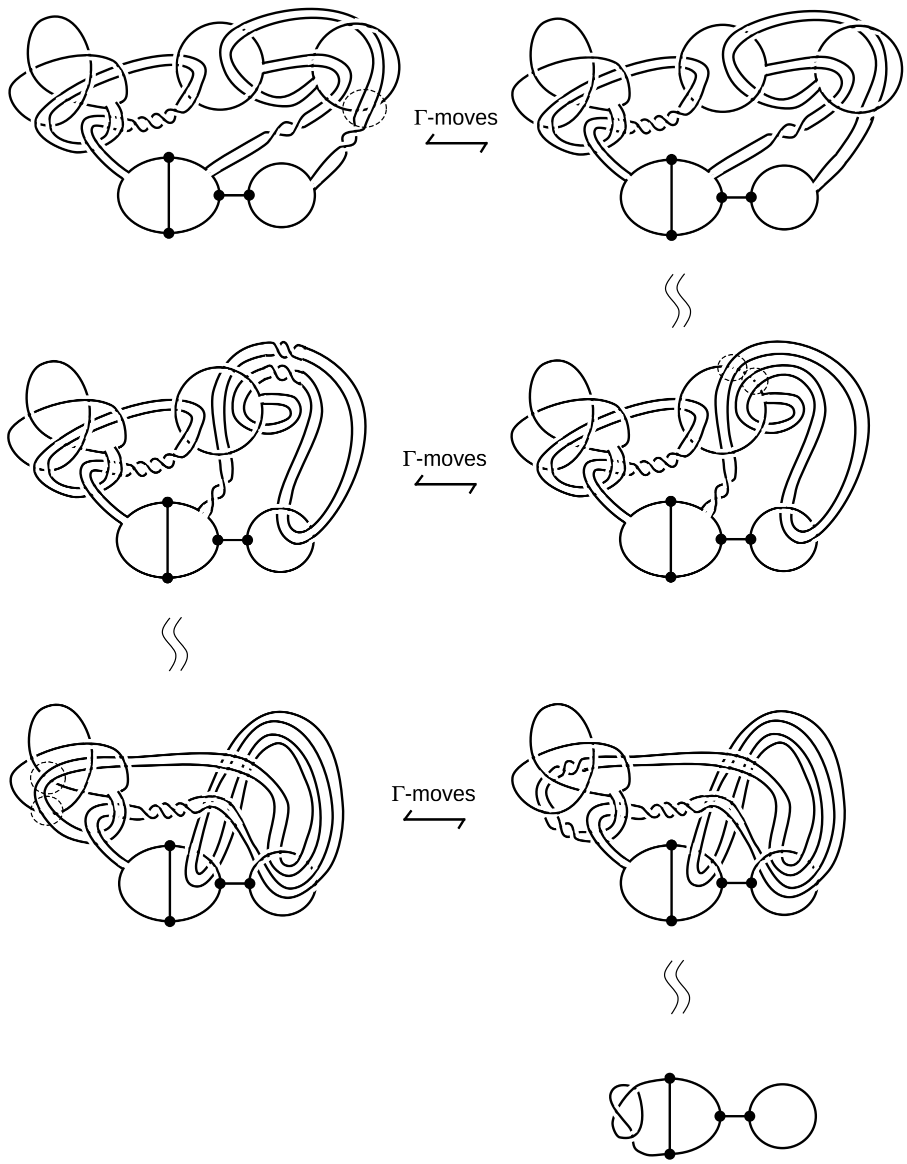

Fig. 3.6.

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