CLASP-PASS MOVES ON LINKS AND HIGHER ORDER COEFFICIENTS OF THE CONWAY POLYNOMIAL

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1. Clasp-pass moves on knots and links

Throughout this report we work in the piecewise linear category and our links are ordered and oriented. K. Habiro introduced a *clasp-pass move* as a local move on links as illustrated in Fig. 1.1 [2]. We call an equivalence relation on links generated by clasp-pass moves and ambient isotopies a *clasp-pass equivalence.* For knots, he showed the following.

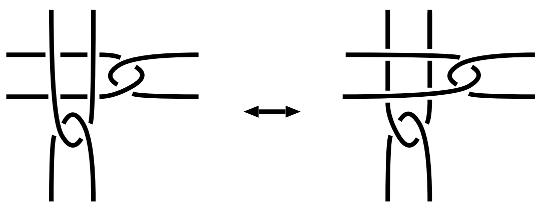

Fig. 1.1.

Theorem 1.1.([3, Proposition 7.1]) *Two knots J and K are clasp-pass equivalent if and only if $a_2(J) = a_2(K)$.* □

Here $a_k(L)$ denotes the k-th coefficient of the *Conway polynomial* of a link L. Namely knots are classified geometrically by this numerical invariant up to clasp-pass equivalence. Besides it is known that if two links L and M are clasp-pass equivalent then $v_2(L) = v_2(M)$ for any *Vassiliev invariant* v_2 of order less than or equal to 2 [1] [3] [8] [9]. The converse is also true for knots (cf. [3, Theorem 1.1]), but not true for n-component links ($n \geq 2$).

We are interested in the question: What invariants do classify n-component links ($n \geq 2$) up to clasp-pass equivalence? K. Taniyama and A. Yasuhara gave an answer for $n = 2, 3$ [10, Theorems 1.5 and 1.7] (see the following table).

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n	invariants
2	linking number lk a_2 of each of components a_3 (mod 2)
3	lk of each of 2-component sublinks a_2 of each of components a_3 (mod 2) of each of 2-component sublinks Milnor invariant μ modulo g.c.d of pairwise linking numbers a_4 (mod 2)
$n \geq 4$	?

We remark here that they give an answer for n-component *algebraically split links* (namely, each of pairwise linking numbers is zero). They are classified by a_2 of each of components, a_3 (mod 2) of each of 2-component sublinks and μ of each of 3-component sublinks [10, Theorem 1.4]. We note that a_3 (mod 2) of a 2-component link, a_4 (mod 2) of a 3-component link and μ modulo the greatest common divisor of pairwise linking numbers of a 3-component link are not Vassiliev invariant of order less than or equal to 2.

Our purpose in this report are to reveal the relationship between the clasp-pass equivalence on links and higher order coefficients of the Conway polynomial. To state our approach, we transform each of links into a specific one up to ambient isotopy. Let $L = J_1 \cup J_2 \cup \cdots \cup J_n$ be a n-component link. We denote $lk(J_i \cup J_j)$ by l_{ij} $(1 \leq i, j \leq n)$. Let $X_{l_{12}l_{13}\cdots l_{n-1,n}} = Y_1 \cup Y_2 \cup \cdots \cup Y_n$ be a n-component link with $lk(Y_i \cup Y_j) = l_{ij}$ $(1 \leq i, j \leq n)$ as illustrated in Fig. 1.2 where the case $n = 4, l_{12} = -1, l_{13} = 3, l_{14} = -2, l_{23} = 2, l_{24} = -2$ and $l_{34} = -1$ is illustrated. A *delta move* is a local move as illustrated in Fig. 1.3. We

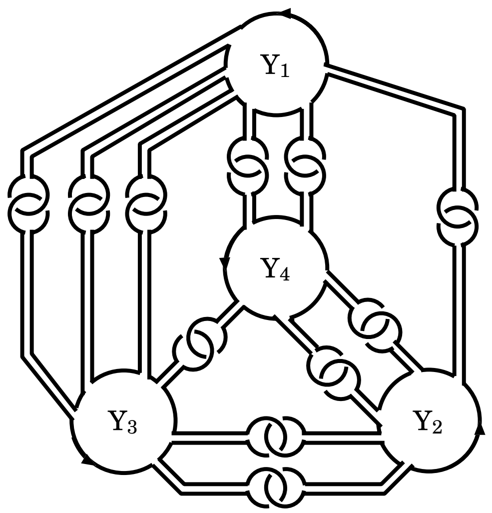


Fig. 1.2.

call an equivalence relation on links generated by delta moves and ambient isotopies a *delta equivalence*. Since it is easy to see that a clasp-pass move is realized by two delta

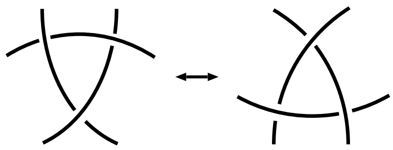


Fig. 1.3.

moves, so we have that a clasp-pass equivalence implies a delta equivalence. It is known that two links are delta equivalent if and only if they have same pairwise linking numbers [7] (see also [10]). Therefore we have the following.

Lemma 1.2. *Two links L and $X_{l_{12}l_{13}\cdots l_{n-1,n}}$ are transformed each other by delta moves and ambient isotopies.* □

Then we can regard a delta move as a *band sum of a Borromean ring* as illustrated in Fig. 1.4. Namely we pretend to apply a delta move to L. In practice we have that

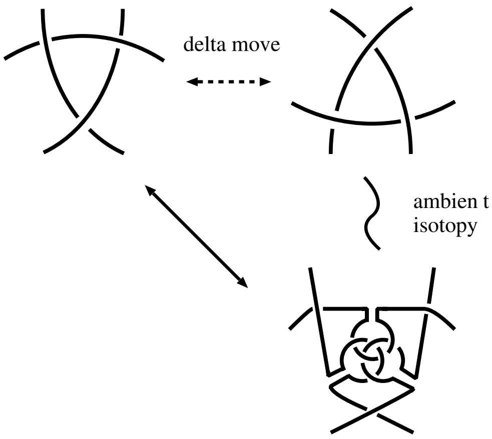


Fig. 1.4.

L is a band sum of Borromean rings and $X_{l_{12}l_{13}\cdots l_{n-1,n}}$ (see [10, Lemma 2.1] for details). We call a locall part as illustrated in Fig. 1.5 a *Borromean chord*. We denote the set of components which has intersection with the chord C by $\varepsilon(C)$. We define that the *type* of C is (i,j,k) if $\varepsilon(C)=\{J_i,J_j,J_k\}$, (i,j) if $\varepsilon(C)=\{J_i,J_j\}$ and (i) if $\varepsilon(C)=\{J_i\}$.

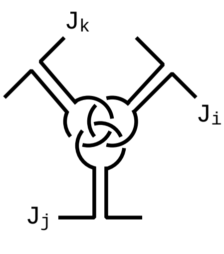


Fig. 1.5. Borromean chord

Now we construct a simple graph F_L as follows. The vertices of F_L are labeled $v_1, v_2, \ldots, v_n$, and v_i and v_j are connected by an edge e_{ij} if l_{ij} is odd. We call F_L a *modulo* 2 *linking graph of* L. Note that F_L is unique up to delta equivalence. We can define a first $\mathbf{Z}_2$-cocycle $\varphi_L \in C^1(F_L; \mathbf{Z}_2)$ by $\varphi_L(e_{ij}) = 0$ if the number of Borromean chords of type (i,j) is even and 1 if the number of Borromean chords of type (i,j) is odd. Then we can state our main results.

Theorem 1.3. *Let $L = J_1 \cup J_2 \cup \cdots \cup J_n$ and $M = K_1 \cup K_2 \cup \cdots \cup K_n$ be delta equivalent n-component links and $F = F_L = F_M$ a modulo 2 linking graph of them. If $a_{m+1}(J_{i_1} \cup J_{i_2} \cup \cdots \cup J_{i_m}) \equiv a_{m+1}(K_{i_1} \cup K_{i_2} \cup \cdots \cup K_{i_m}) \pmod 2$ for $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ and $3 \leq m \leq n$, then $[\varphi_L] = [\varphi_M]$ in $H^1(F; \mathbf{Z}_2)$.*

We note that each of $a_{m+1}(J_{i_1} \cup J_{i_2} \cup \cdots \cup J_{i_m}) \pmod 2$ for $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ and $3 \leq m \leq n$ for an n-component link $L = J_1 \cup J_2 \cup \cdots \cup J_n$ is an invariant under a clasp-pass equivalence [10, Lemmas 2.6 and 2.7]. Thus as a corollary of Theorem 1.3, we have the following.

Corollary 1.4. *Let $L = J_1 \cup J_2 \cup \cdots \cup J_n$ and $M = K_1 \cup K_2 \cup \cdots \cup K_n$ be n-component links. If L and M are clasp-pass equivalent, then $[\varphi_L] = [\varphi_M]$ in $H^1(F; \mathbf{Z}_2)$, where $F = F_L = F_M$ is a modulo 2 linking graph of them.* □

This invariant play an important role for classification of links up to clasp-pass equivalence.

2. Idea of the proof

Lemma 2.1. ([10, Lemma 2.5]) *Each pair of the embeddings illustrated in Fig. 2.1 and 2.2 are clasp-pass equivalent.* □

Specially, the pair of the embeddings illustrated in Fig. 2.3 are clasp-pass equivalent [10]. By using deformations above, we can deform L up to clasp-pass equivalence so that

(1) each Borromean chord of type (i) is contained in a 3-ball as illustrated in Fig. 2.4 (a) or (b), and for each i, not both of (a) and (b) occur,

(2) each Borromean chord of type (i,j) is contained in a 3-ball as illustrated in Fig. 2.4 (c), and for $1 \leq i < j \leq n$ there is at most one Borromean chord of type (i,j) and

(3) each Borromean chord of type (i,j,k) is contained in a 3-ball as illustrated in Fig. 2.4 (d) or (e), and for each i,j,k, not both of (d) and (e) occur.

For Borromean chords of type (i), we can see easily that each Borromean chord as illustrated in Fig. 2.4 (a) and (b) is regarded as a connected sum of a trefoil knot and a connected sum of a figure eight knot, respectively. Then the (signed) number of Borromean chords of type (i) coincides with $a_2(J_i)$. For Borromean chords of type (i,j), in fact the number of Borromean chord of type (i,j) coincides with $a_3(J_i \cup J_j) \pmod 2$ if l_{ij} is even. For Borromean chords of type (i,j,k), it is known that the number of Borromean chords can be estimated by $\mu(J_i \cup J_j \cup J_k)$ for specific cases.

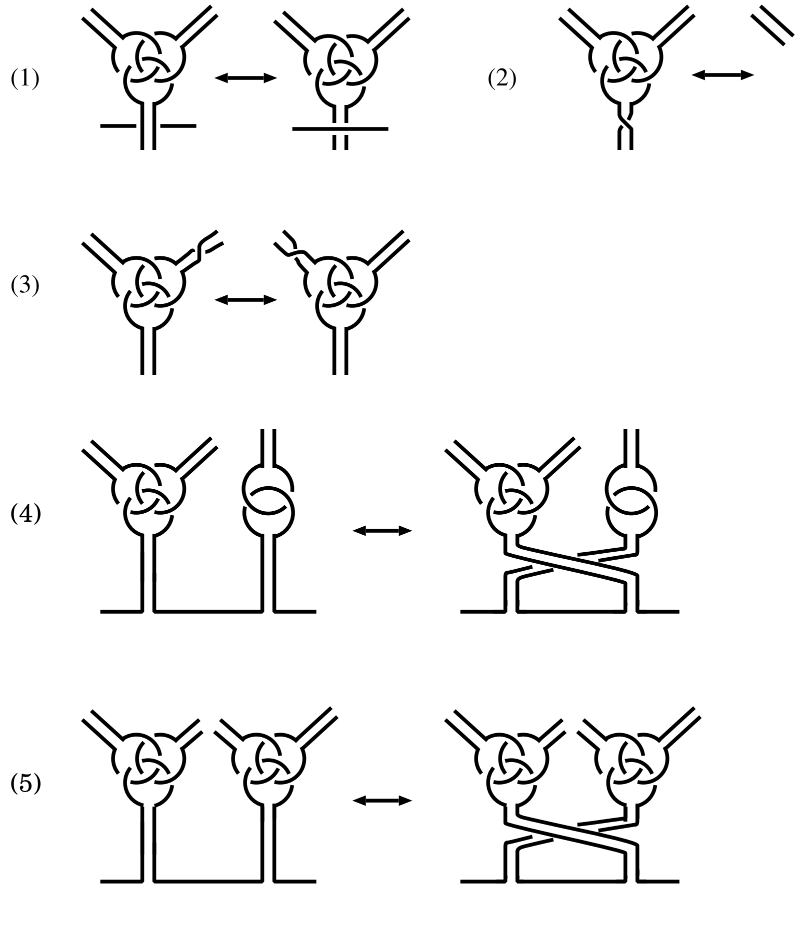
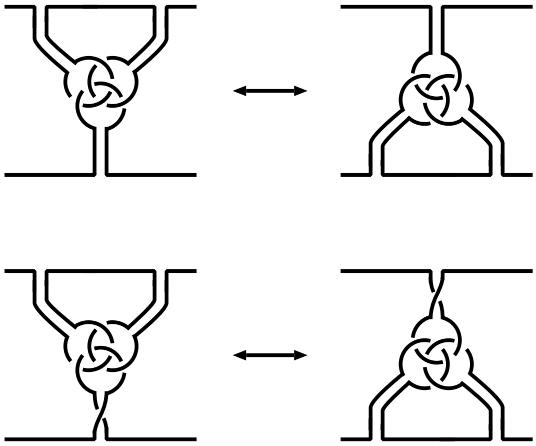


Fig. 2.1.

(6)



(7)

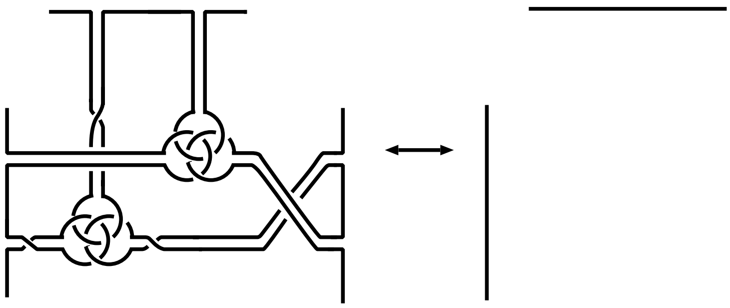


Fig. 2.2.

(8)

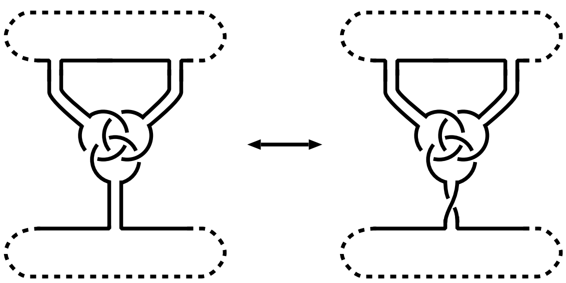


Fig. 2.3.

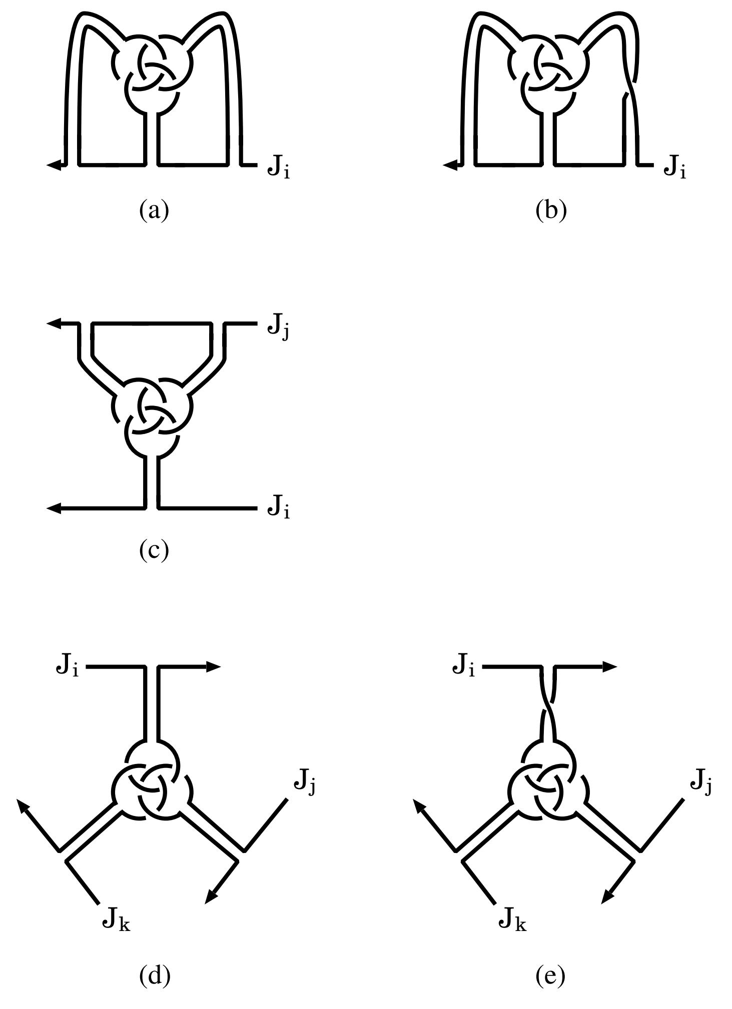


Fig. 2.4.

In fact our invariant $[\varphi_L] \in H^1(F_L; \mathbf{Z})$ can get control the Borromean chords of type (i,j) with odd l_{ij}. For $X_{l_{12}l_{13}\cdots l_{n-1,n}}$, we can create the Borromean chords of type (i,j) if $l_{ij} \neq 0$ as in Fig. 2.5. Since we can create a full-twist on each of Hopf bands around J_i

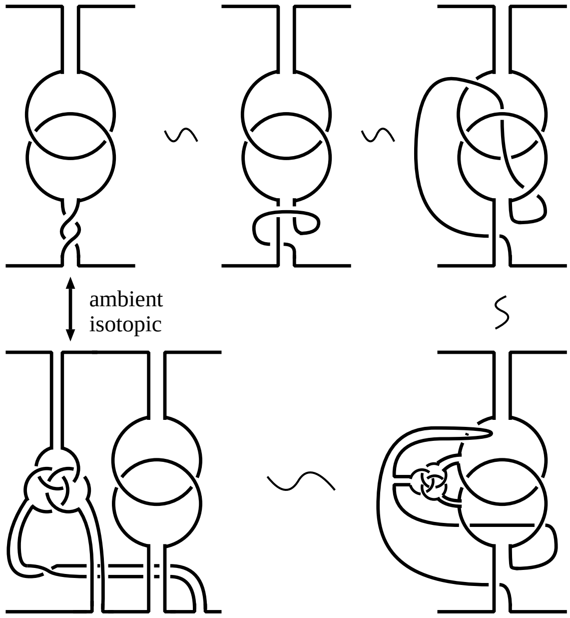


Fig. 2.5.

up to clasp-pass equivalence by turning J_i twice (see Fig. 2.6), for any i we can create l_{ij} Borromean chords of type (i,j) $(1 \leq j \leq n, i \neq j)$ up to clasp-pass equivalence. Note that the above deformations have an influence on φ_L but have no influences on $[\varphi_L]$ because this change is absorbed by the coboundary relations. Let $F^{(q)}$ be a connected component of F_L $(1 \leq q \leq \omega)$. Let T_q be a spanning tree of $F^{(q)}$. For a graph G, we denote the edge set of G by $E(G)$. We note that each of edges in $\mathcal{B}_q = E(F^{(q)}) - E(T_q)$ $(1 \leq q \leq \omega)$ represents a basis of $H^1(F_L; \mathbf{Z}_2)$. Then by succesive applications of Fig. 2.6 along T_q, we can replace all Borromean chords of type (i,j) with odd l_{ij} by Borromean chords of type (i',j') for $e_{i'j'} \in \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_\omega$. Note that by further applications of clasp-pass moves we have that each Borromean chord of type (i',j') for $e_{i'j'} \in \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_\omega$ is contained in a 3-ball as illustrated in Fig. 2.4 (c), and there is at most one Borromean chord of type (i',j'). The above deformations do not have an influence on φ_L. We denote this 'canonical type' got from L by $L' = J'_1 \cup J'_2 \cup \cdots \cup J'_n$ (ordering of components has been preserved from L). As we noted above, we have that $[\varphi_L] = [\varphi_{L'}]$. Let $\gamma_{i'j'} = v_{i'}v_{k_1}v_{k_2}\cdots v_{k_l}v_{j'}v_{i'}$ be the basis of $H_1(F; \mathbf{Z}_2)$ represented by $e_{i'j'}$. We denote the corresponding sublink of L' to $\gamma_{i'j'}$ by $L'(\gamma_{i'j'})$ and the number of components by m $(3 \leq m \leq n)$ (see Fig. 2.7, where Borromean chords are not illustrated). We may assume that $L'(\gamma_{i'j'})$ is one of L_1 or L_2 as illustrated in Fig. 2.8. By using the skein relation at the marked crossing point in Fig. 2.8 and J. Hoste's result [4], we have that

$$a_{m+1}(L_2) - a_{m+1}(L_1) = a_m(L_3) = a_m(L_4) + a_{m-1}(L_5) = a_{m-1}(L_5)$$

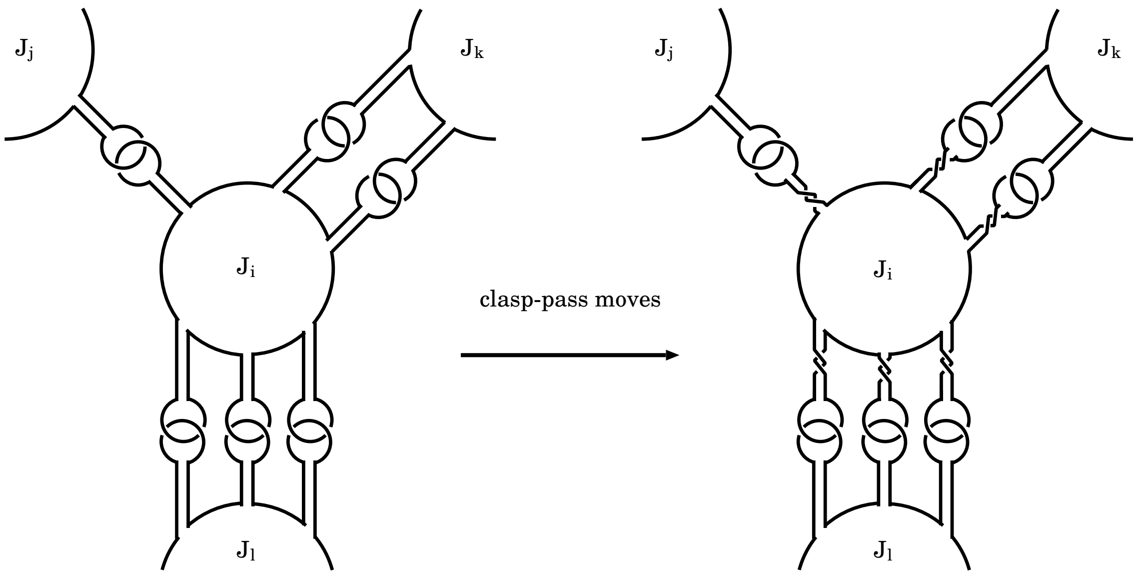

J_j
J_k
J_i
J_l
clasp-pass moves
J_j
J_k
J_i
J_l

Fig. 2.6.

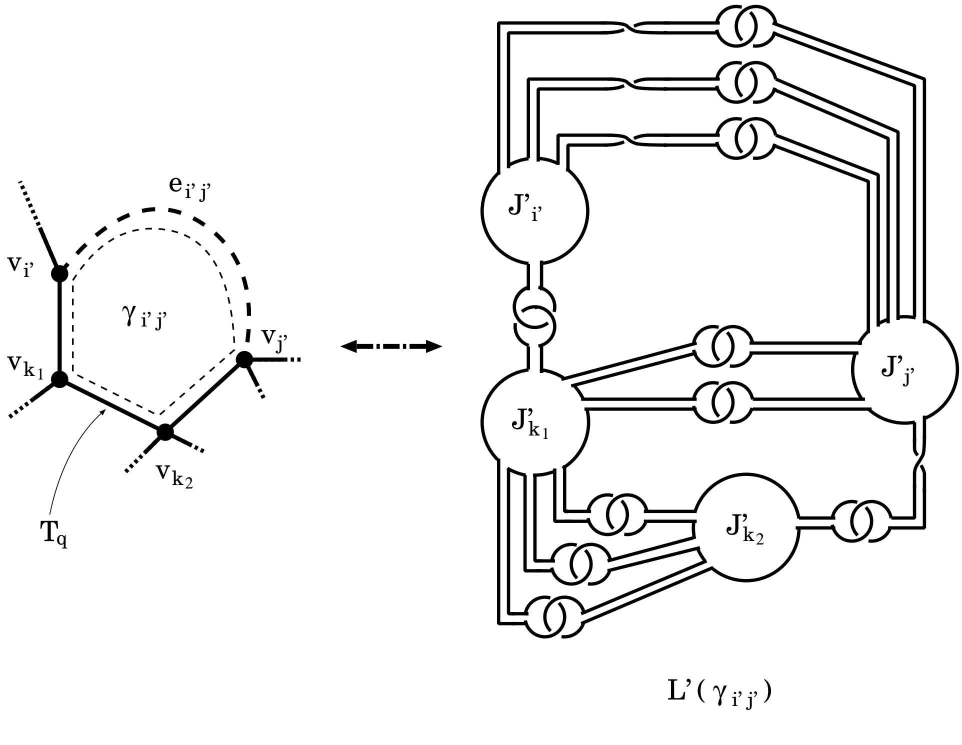

$e_{i'j'}$
$v_{i'}$
$\gamma_{i'j'}$
$v_{j'}$
v_{k_1}
v_{k_2}
T_q
$J'_{i'}$
$J'_{j'}$
J'_{k_1}
J'_{k_2}
$L'(\gamma_{i'j'})$

Fig. 2.7.

$$\equiv l_{i'k_1} l_{ik_1k_2} \cdots l_{k_{m-2}j'} \equiv 1 \pmod 2.$$

This shows that we can check that the modulo 2 parity of $a_{m+1}(L'(\gamma_{i'j'}))$ changes under

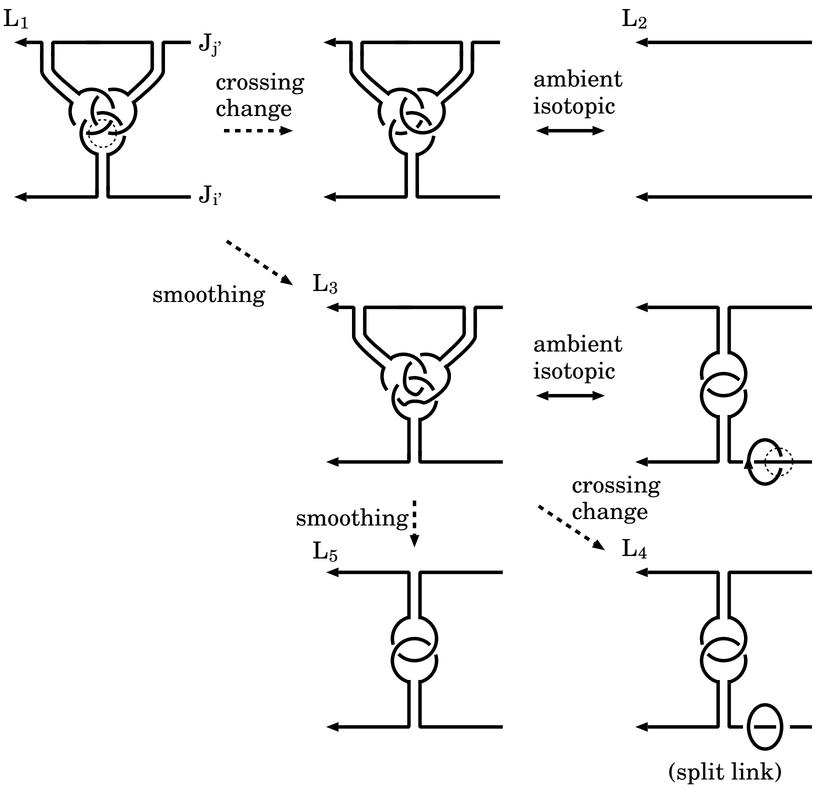


Fig. 2.8.

a band sum of a Borromean chord of type (i', j'). This implies the proof of Theorem 1.3.

Let $L = J_1 \cup J_2 \cup \cdots \cup J_n$ and $M = K_1 \cup K_2 \cup \cdots \cup K_n$ be n-component delta equivalent links and F a modulo 2 linking graph of them. Then we can prove that if $[\varphi_L] = [\varphi_M]$ in $H^1(F; \mathbf{Z}_2)$ then we can deform the Borromean chords of type (i, j) with odd l_{ij} for L and M identically up to clasp-pass equivalence. So we can control Borromean chords of type (i, j) with odd l_{ij} completely.

3. Some classifications

For an n-component link $L = J_1 \cup J_2 \cup \cdots \cup J_n$, we construct a simple graph G_L as follows. Let $\{v_1, v_2, \ldots, v_n\}$ be the set of vertices of G_L, and v_i and v_j are connected by an edge $e_{ij} = v_iv_j$ if $lk(J_i \cup J_j) \neq 0$. We call this graph G_L a *linking graph* of L. A link L is said to be *acyclic* if G_L is a forest, and *cyclic* if G_L is a n-cycle, namely which contains exactly n vertices. We note that an algebraically split link is acyclic. Then we have the following classification theorems for links up to clasp-pass equivalence.

Theorem 3.1. *Let $L = J_1 \cup J_2 \cup \cdots \cup J_n$ and $M = K_1 \cup K_2 \cup \cdots \cup K_n$ be acyclic n-component links. Then L and M are clasp-pass equivalent if and only if the following*

conditions hold;
(1) $lk(J_i \cup J_j) = lk(K_i \cup K_j)$ *for* $1 \leq i < j \leq n$,
(2) $a_2(J_i) = a_2(K_i)$ *for* $1 \leq i \leq n$,
(3) $a_3(J_i \cup J_j) \equiv a_3(K_i \cup K_j) \pmod 2$ *for* $1 \leq i < j \leq n$ *and*
(4) $\mu(J_i \cup J_j \cup J_k) \equiv \mu(K_i \cup K_j \cup K_k)$ *modulo the greatest common divisor of* $lk(J_i \cup J_j)$, $lk(J_j \cup J_k)$ *and* $lk(J_i \cup J_k)$ *for* $1 \leq i < j < k \leq n$.

Theorem 3.2. *Let* $L = J_1 \cup J_2 \cup \cdots \cup J_n$ *and* $M = K_1 \cup K_2 \cup \cdots \cup K_n$ *be cyclic* n*-component links. Then* L *and* M *are clasp-pass equivalent if and only if the following conditions hold;*
(1) $lk(J_i \cup J_j) = lk(K_i \cup K_j)$ *for* $1 \leq i < j \leq n$,
(2) $a_2(J_i) = a_2(K_i)$ *for* $1 \leq i \leq n$,
(3) $a_3(J_i \cup J_j) \equiv a_3(K_i \cup K_j) \pmod 2$ *for* $1 \leq i < j \leq n$,
(4) $\mu(J_i \cup J_j \cup J_k) \equiv \mu(K_i \cup K_j \cup K_k)$ *modulo the greatest common divisor of* $lk(J_i \cup J_j)$, $lk(J_j \cup J_k)$ *and* $lk(J_i \cup J_k)$ *for* $1 \leq i < j < k \leq n$ *and*
(5) $a_{n+1}(L) \equiv a_{n+1}(M) \pmod 2$.

Since any 2-component links and algebraically split links are acyclic, and any 3-component links are acyclic or cyclic, we have the classifications of $2, 3$-component links and algebraically split links as corollaries of Theorems 3.1 and 3.2.

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