On Heegaard genus, bridge genus and braid genus for a 3-manifold

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We introduce the bridge genus and the braid genus as two kinds of genera of a closed connected orientable 3-manifold, and compare them with the Heegaard genus. We consider four inequalities between these three genera and we construct a 3-manifold which satisfies each one of inequalities.

Let $M$ be a closed connected orientable 3-manifold. Then there exist handlebodies $H_1$ and $H_2$ of same genus and a homeomorphism $f : \partial H_1 \to \partial H_2$ such that $M = H_1 \cup_f H_2$. We call the triple $(H_1, H_2; f)$ a Heegaard splitting of $M$ and we call $f(\partial H_1) = \partial H_2$ the Heegaard surface. The Heegaard genus of $M$ is the minimal genus of Heegaard surfaces, denoted by $g_H(M)$.

Let $L = K_1 \cup K_2 \cup \cdots \cup K_n$ be an $n$-component link in the 3-sphere $S^3$, and $N(L)$ a tubular neighborhood of $L$, and $E(L)$ the exterior $S^3 - N(L)$ of $L$. Here for a set $Y$ in a topological space $X$, $\overline{Y}$ is the closure of $Y$ in $X$. Let $\chi(L, 0)$ be the 3-manifold obtained from $E(L)$ by attaching $n$ solid tori $V_1, V_2, \ldots, V_n$ to $\partial E(L)$ such that the meridian of $\partial V_i$ is mapped to the longitude of $K_i$ ($1 \leq i \leq n$).

The bridge genus $g_{bridge}(M)$ (resp. the braid genus $g_{braid}(M)$) of $M$ is the minimal number of bridge($L$) (resp. braid($L$)) for any $L$ such that $M$ is obtained by the 0-surgery of $S^3$ along $L$. The bridge genus and the braid genus are introduced by A.Kawauchi.

**Theorem 0.1.** Let $M$ be a closed connected orientable 3-manifold. Then we obtain

$$g_H(M) \leq g_{bridge}(M) \leq g_{braid}(M).$$

**Example 0.2.** Let $M$ be the connected sum of $n$ copies of $S^1 \times S^2$. Then

$$g_H(M) = g_{bridge}(M) = g_{braid}(M) = n.$$ 

**Example 0.3.** For the Hopf link $L$, we have $\chi(L, 0)$ is homeomorphic to $S^3$. Then

$$0 = g_H(S^3) < g_{bridge}(S^3) = g_{braid}(S^3) = 2.$$ 

**Example 0.4.** Let $K$ be the figure-eight knot, and $M = \chi(K, 0)$. Then

$$2 = g_H(M) = g_{bridge}(M) < g_{braid}(M) = 3.$$ 

**Example 0.5.** Let $M = \chi(8_{15}, 0)$. Then

$$g_H(M) = 2 < g_{bridge}(M) = 3 < g_{braid}(M) = 4.$$